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Focus so far:

(informal) Polynomial Time

P = problems that can be solved in $O(n^k)$ time, k a constant, if description of input is size n (bits)

$k=10^6$ not very efficient... but almost all problems in P actually have $k=1,2,3,4$. (like most problems in class)

e.g. Adjacency list: Size: $O((n+m)\log n)$ $\Rightarrow$ graph alg run time is $O(n+m)$, $O(nm)$, $O(n\log m)$

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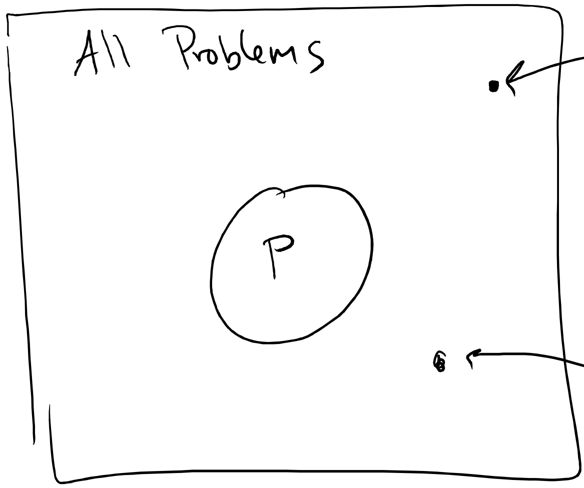

Hamiltonian Path (probably not in P, but can solve in exponential time)

would like to distinguish between these

Halting Problem (no computer can figure out, even with unlimited resources)

Motivates: Nondeterministic Polynomial Time

NP (informal) = set of problems where if input is size n,

- Solution is size $O(n^{k_1})$
- Can verify if solution is correct in $O(n^{k_2})$ time

ex: Hamiltonian Path

Input: Adjacency List of directed, unweighted graph $G=(V,E)$; $s,t \in V$. $|V|=n$, $|E|=m$

Output: If it exists, a path from s to t that goes through each vertex once.

Another NP problem:

<u>3SAT</u>: Given CNF formula of $x_1, x_2, \dots x_n$ and negations

$\underset{\text{AND of ORs}}{\uparrow}$

where each clause has at most 3 terms, is there a satisfying assignment?

Example input:

$$\underbrace{(x_1 \vee x_3 \vee \overline{x_4})}_{C_1} \wedge \underbrace{(\overline{x_1} \vee \overline{x_2} \vee \overline{x_5})}_{C_2} \wedge \underbrace{(\overline{x_3} \vee x_5)}_{C_3}$$

Clauses: C_1 C_2 C_3

Output: $x_1 = 1$, $x_2 = 0$, ...

Most people believe NP-Hard problems take exponential time to solve

Verify HAMPATH $\in$ NP

- Size of Solution:

$$(n-1)\cdot \log n$$

↑ # of vertices in path

↖ vertex nam

- Time to verify:

$O(n^2)$ ← · checking each edge is valid takes time $O(n)$
· need to do n times

$+$ $O(n)$ ← · maintain array of visited vertices to check all visited exactly once.

HAMPATH $\in$ NP

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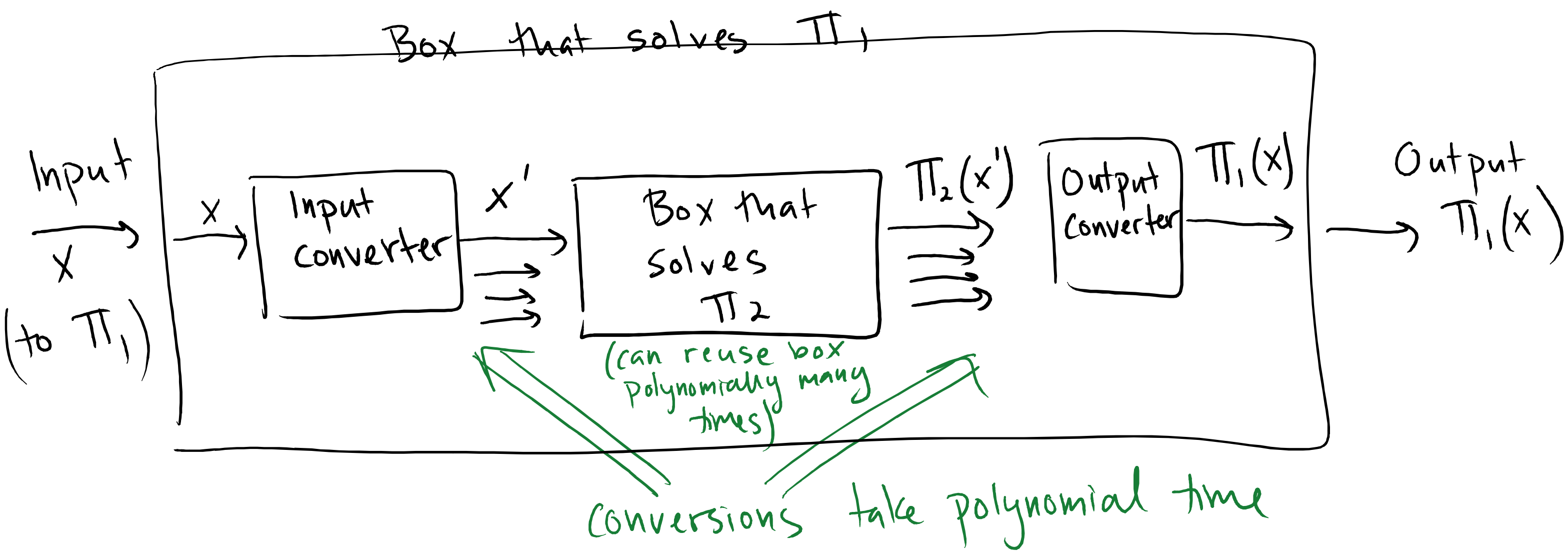
Hamitonian cycle is hardest problem in NP

what does this mean? It can be used to solve any other problem in NP

Polynomial Reduction

Problem Π_1 reduces to problem Π_2 if



To show 3SAT reduces to HAMPATH:

→ X = CNF formula

→ Input converter uses X to create X', an Adj. List

→ Output HAMPATH(X') tell you if Ham path exists

→ Output converter uses this info to decide if CNF formula is satisfiable

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Q: If $\Pi_2 \in P$ and Π_1 reduces to Π_2 then

A) $\Pi_1 \in P$

B) $\Pi_1 \notin NP$

C) We don't have enough information

$\Pi_1 \in P$ because need to run Π_2 poly # of times, takes poly time, and do poly time other steps $\Rightarrow$ poly time.

Π_1 reduces to Π_2 implies Π_2 is harder than Π_1 (if can do Π_2, can do Π_1)

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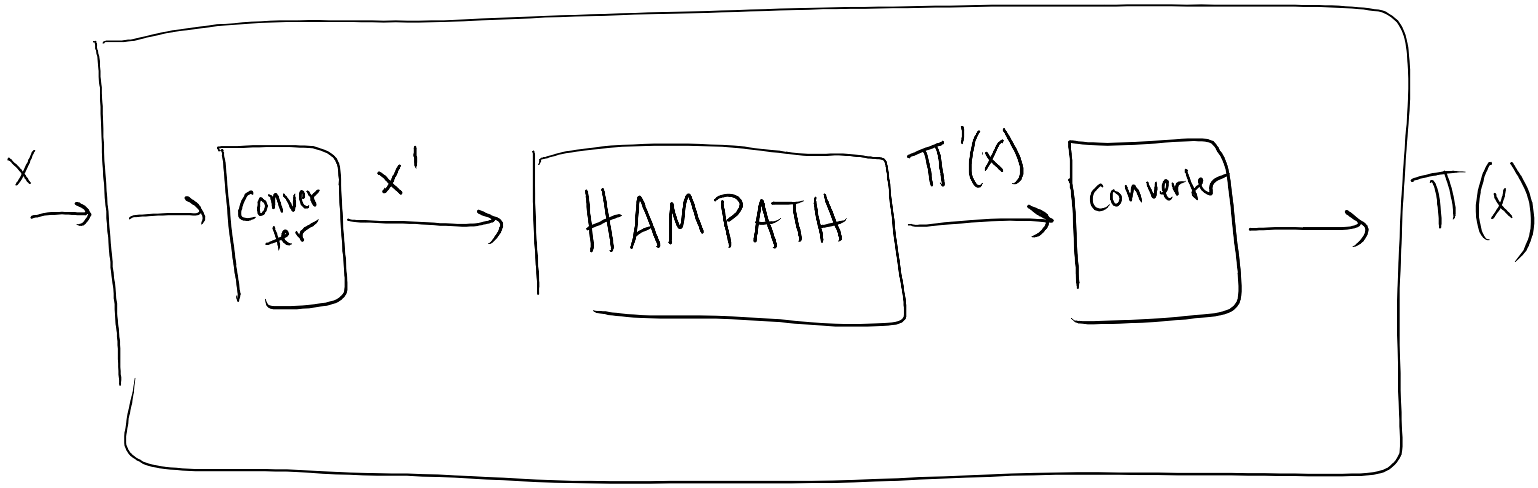
Can show $\forall \Pi \in NP$, Π reduces to HAMPATH



- HAMPATH is NP-Hard (every problem in NP reduces to it)
- Also HAMPATH ∈ NP

$\Downarrow$

HAMPATH is NP-Complete

$$\boxed{\Pi \in NP \wedge \Pi \in \text{NP-Hard} = \text{NP-COMPLETE}}$$

We will take as fact that 3SAT is NP-COMPLETE

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Important Skill in algorithm design proving Π is NP-Hard

Why?

- Won't spend time trying to find efficient solution
- Look up existing alg for NP-hard problems
- Use special structure (average case might be easy)

$\boxed{\text{Strategy for proving } \Pi \text{ is NP-Hard}} \Rightarrow$ Reduce 3SAT to Π

(Π harder than 3SAT) & (3SAT harder than NP)

$\Rightarrow$ Π is harder than NP

<u>Prove</u>: 3SAT reduces to HAMPATH

1) Create X' (input to HAMPATH) using X (input to 3SAT)

2) Show takes poly time to do reduction

3) Show 3SAT(X) has solution iff HAMPATH(X') has solution

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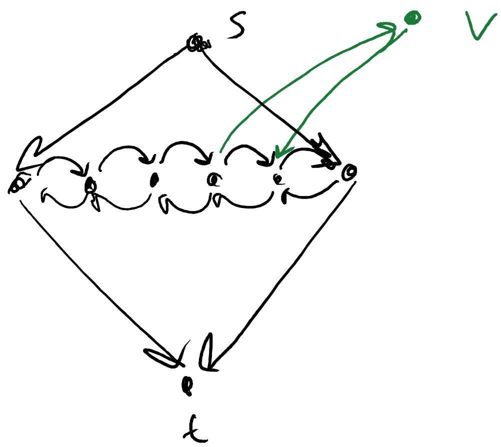

Q: How many Hamiltonian Paths are there from s to t without v / with v?

A) 1, 0

B) 1, 1

C) 1, 2

D) 2, 1 ⟸