CS302 - Problem Set 5
Due: Monday, Oct 16. Must be uploaded to Canvas before the beginning of class.

1. (*) Suppose you have a line graph on 6 vertices with the following weights:

$$w(v_1) = 3 \qquad w(v_2) = 7 \qquad w(v_3) = 10 \qquad w(v_4) = 5 \qquad w(v_5) = 4 \qquad w(v_6) = 3$$

   (a) [**3 points**] Let A_i be the weight of the maximum independent set on the graph G_i (where G_i is the line graph on the first i vertices with the same weights as above). Give the values of A_i for $i = 0$ to $i = 6$.

   (b) [**3 points**] What is the maximum weight independent set for G_6? (Show roughly how you work backwards by getting expressions for the MWIS on G_i in terms of the MWIS of G_j for $j < i$.)

2. (**) Suppose you are hosting a music festival with multiple stages, and you are trying to decide which of n possible groups should perform at your prime 8pm Saturday slot. The groups would perform at the same time but on different stages (and assume you have unlimited stages). To figure out which bands should play, you give a survey to 100 early-bird registrants, and have them rank the bands that they are most excited to see at the festival. Assume each band will only perform once over the course of the festival.

   (a) [**6 points**] Describe how you could use the algorithm for Max Weight Independent Set to choose which groups should perform in the prime slot.

   (b) [**6 points**] If you use the approach from part (a), in what sense would your selection of bands be optimal?

   (c) [**3 points**] Now that you've figured out who should perform on Saturday night, if the second prime time slot is at 8pm on Friday, how should you pick which bands should perform then using MWIS?

3. (**) For each of the following statements, if it is true, prove it, and if it is false, provide a proof by counterexample.

   (a) [**11 points**] If a path graph has at least two vertices, the minimum-weight vertex is never part of the maximum-weight independent set.

   (b) [**11 points**] Considering the dynamic programming algorithm for MWIS on a path graph, if a vertex is excluded from the optimal solution of two consecutive sub-problems, then it is excluded from the optimal solution to the final problem.

4. Prove the correctness of the dynamic programming algorithm for MWIS. (These questions are both relatively straightforward applications of our favorite proof technique and results from class. If you feel like you don't need the practice, then feel free to skip.)

   (a) [**0 points**] Prove that our construction of the array A correctly sets $A[i]$ equal to the weight of the maximum independent set on G_i.

(b) [**0 points**] Prove that our approach for working backwards to determine the maximum independent set on G_n using A is correct.

5. (***) [**11 points**] Describe an algorithm that takes as input an unsorted array of unique integers of length n (you may assume n is a power of 2), and outputs the *second* largest value in the array. Prove that your algorithm is correct, and only uses $n + \log_2 n - 2$ comparison operations, where a comparison is an operation that tests whether one element is less than another element. (Hint: the solution involves ideas from both Divide and Conquer, and Dynamic Programming.)