CS302 - Problem Set 0

1. Prove that the algorithm is correct.

Solution Let $P(n)$ be the predicate that `RecMultiplication` correctly multiplies two n-digit numbers, for $n \geq 1$. We prove correctness by strong induction on n.

For the base case, if $n = 1$, the algorithm triggers the base case and returns $a * b$, so $P(n)$ is true.

Now for the inductive step. For strong induction, we assume that $P(k)$ is true for all k such that $n > k \geq 1$. Using this assumption, we will prove $P(n)$ is true. Since $n > 1$, the recursive case of the algorithm triggers. Note that each of the recursive calls in line 14 involves a multiplication of two numbers with $n - h$ digits (thanks to our padding with zero steps), where $h = \lfloor n/2 \rfloor \geq 1$. Since $n - h < n$, we may assume the algorithm returns the correct output on these inputs. That is, it returns the product of the two inputs.

Therefore, the algorithm returns

$$\begin{aligned}
&10^{2h}\left(\sum_{i=0}^{n-h-1} a_i^1 10^i\right)\left(\sum_{j=0}^{n-h-1} b_j^1 10^j\right) + \left(\sum_{i=0}^{n-h-1} a_i^0 10^i\right)\left(\sum_{j=0}^{n-h-1} b_j^0 10^j\right) \\
&+ 10^h\left(\sum_{i=0}^{n-h-1} a_i^0 10^i\right)\left(\sum_{j=0}^{n-h-1} b_j 10^j\right) + 10^h\left(\sum_{i=0}^{n-h-1} a_i^1 10^i\right)\left(\sum_{j=0}^{n-h-1} b_j^0 10^j\right) \\
=&\left(10^h \sum_{i=0}^{n-h-1} a_i^1 10^i + \sum_{i=0}^{n-h-1} a_i^0 10^i\right)\left(10^h \sum_{j=0}^{n-h-1} b_j^1 10^j + \sum_{j=0}^{n-h-1} b_j^0 10^j\right) \\
=&a \times b. \qquad (1)
\end{aligned}$$

Thus, by strong induction, the algorithm is correct for all $n \geq 1$.