Name ___________________

CS200 - Midterm 1 Review Questions

1. **[11 points]** Prove using induction that program `ProductThing`(n) returns a number less than or equal to n^n for all $n \geq 1$.

Algorithm 1: `ProductThing`(m)

Input : $m \in \mathbb{Z}$ such that $m \geq 1$
Output: Something.
`// Base Case`
1 **if** $m == 1$ **then**
2   return 1;
3 **else**
    `// Recursive step`
4   return `ProductThing`($m-1$) $\times m$
5 **end**

Solution Let $P(n)$ be the predicate that `ProductThing`(n) returns a number less than or equal to n^n. We will prove using induction that $P(n)$ is true for all $n \geq 1$.

Base case: $P(1)$ is true because when the input to the algorith is 1, the algorithm returns 1 at line 2, which is equal to 1^1.

Inductive step: Let $k \geq 1$. Assume for induction that $P(k)$ is true. Let's consider what happens when the input to `ProductThing` is $k+1$. Then since $k+1 > 1$, we go to Line 4 and return `ProductThing`$(k+1-1) \times (k+1) =$ `ProductThing`$(k) \times (k+1)$. From our inducttive assumption, we know `ProductThing`$(k) \leq k^k$. Multiplying both sides of this inequality by $(k+1)$, we have

$$\texttt{ProductThing}(k) \times (k+1) \leq k^k \times (k+1). \tag{1}$$

Now since $k \geq 1$, we know $k^k \leq (k+1)^k$. Multiplying both sides of this inequality by $(k+1)$ we have $k^k(k+1) \leq (k+1)^k \times (k+1) = (k+1)^{k+1}$. Thus (using the transitive property of inequalities)

$$\texttt{ProductThing}(k) \times (k+1) \leq (k+1)^{k+1}. \tag{2}$$

Therefore $P(k+1)$ is true.

Thus, $P(n)$ is true for all $n \geq 1$.

2. Let S be the set of all people who have every lived. Let $G(x, y)$ be the predicate, x is the grandmother of y, for $x, y \in S$. Let $C(x, y)$ be the predicate x and y are cousins for $x, y \in S$. Write the following statements and predicates using math:

   (a) "All people have at least two grandmothers."

(b) "Every pair of cousins share a grandmother."

(c) "None of Person x's cousins are grandmothers "

(d) (Challenge) "All of Person x's children except one are childless."

Solution

(a) $\forall x \in S, \exists y, z \in S : (y \neq z) \wedge G(y, x) \wedge G(z, x)$.

(b) $\forall x, y \in S, C(x, y) \rightarrow (\exists z \in S : G(z, x) \wedge G(z, y))$.

(c) $\forall y \in S, C(x, y) \rightarrow (\neg \exists w \in S : G(y, w))$.

(d) $(\exists y \in S : G(x, y)) \wedge (\neg \exists w \in S : C(w, y))$.

3. You meet a group of 50 orcs. You know orcs are either honest or corrupt. Suppose you know that at least one of the orcs is honest. You also know that given any two of the orcs, at least one is corrupt. Let G be the set of orcs, and $D(g)$ is the predicate "orc g is corrupt."

   (a) How many of the orcs are corrupt and how many are honest?

   (b) Express "At least one orc is honest" using math.

   (c) Express "Given any two orcs, at least one is corrupt" using math

   Solution

   (a) There is one honest orc. If there were two honest orcs then you could have a pair where both are honest, and we know that any pair has to have a corrupt orc.

   (b) $\exists g \in G : \neg D(g)$.

   (c) $\forall x, y \in G, x \neq y \rightarrow (D(x) \vee D(y))$.

4. (This is a little harder than I would ask on an exam, but good practice!) Prove that $1/1 + 1/4 + 1/9 + \cdots + 1/n^2 \leq 2 - 1/n$ for all $n \geq 1$, .

   Solution Let $P(n)$ be the predicate that $1/1+1/4+1/9+\cdots+1/n^2 \leq 2-1/n$. We will prove $P(n)$ for all $n \geq 1$ using induction.

   Base case: $P(1)$ is true because $1/1^2 = 2 - 1/1 = 1$.

   Inductive step: Let $k \geq 1$. Assume for induction that $P(k)$ is true. That means

$$1/1 + 1/4 + 1/9 + \cdots + 1/k^2 \leq 2 - 1/k. \tag{3}$$

   If we add $1/(k+1)^2$ to both sides, we get

$$1/1 + 1/4 + 1/9 + \cdots + 1/k^2 + 1/(k+1)^2 \leq 2 - 1/k + 1/(k+1)^2. \tag{4}$$

Let's add $0 = -1/(k+1) + 1/(k+1)$ to the right hand side to get the right hand side to look more the way we want. Then if we simplifying the extra terms, we have

$$\begin{aligned} 2 - \frac{1}{k} + \frac{1}{(k+1)^2} &= 2 - \frac{1}{k+1} + \left[\frac{1}{k+1} - \frac{1}{k} + \frac{1}{(k+1)^2}\right] \\ &= 2 - \frac{1}{k+1} + \frac{k(k+1) - (k+1)^2 + k}{k(k+1)^2} \\ &= 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2} \\ &< 2 - \frac{1}{k+1}, \end{aligned} \tag{5}$$

where in the last line, we've used the fact that $-\frac{1}{k(k+1)^2} < 0$ when $k \geq 1$. Thus using the transitive property of equality and inequality, we have $P(k+1)$ is true.

Thus, $P(n)$ is true for all $n \geq 1$.