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Learning Goals

- Properly use multiple base cases in a strong inductive proof.
- Describe graphs using math language
- Describe some real world graph applications.

$F(n)$

1. If $n \le 1$: return n
2. return $5 \cdot F(n-1) - 6 \cdot F(n-2)$

Q: Prove this algorithm returns $3^n - 2^n$ for all $n \ge 0$.

Let $P(n)$ be the predicate $F(n)$ returns $3^n - 2^n$. We will prove $P(n)$ is true for all $n \ge 0$, using strong induction

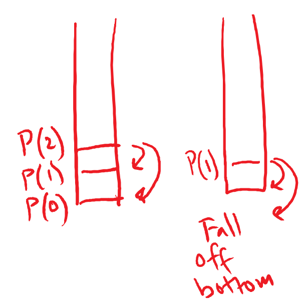
Base cases: We will show $P(0)$ and $P(1)$. When the input is 0, we return 0. Since $3^0 - 2^0 = 1 - 1 = 0$, this is correct. When the input is 1, we return 1. Since $3^1 - 2^1 = 3 - 2 = 1$, this is correct.

Inductive step: Let $\boxed{k \ge 1}$. Assume $P(j)$ is true for all j such that $\boxed{0 \le j \le k}$. We will prove $P(k+1)$

We want $k+1$ to be larger than base cases, so choose k to be larger than or equal to largest base case

Want to assume all base cases are true, so j starts at smallest base case.

We need to prove $P(0)$ and $P(1)$. Otherwise when try to prove $P(2)$, look at $f(2-1) = f(1)$ and $f(2-2) = f(0)$, need to assume these output correctly



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F(n)
1. If $n \le 1$: return n
2. return $5 \cdot F(n-1) - 6 \cdot F(n-2)$

Let P(n) be the predicate F(n) returns $3^n - 2^n$. We will prove P(n) is true for all $n \ge 0$, using strong induction

Base case: There are two base cases: when the input is 0, we return 0. Since $3^0 - 2^0 = 1 - 1 = 0$, this is correct. When the input is 1, we return 1. Since $3^1 - 2^1 = 3 - 2 = 1$, this is correct.

Inductive step: We assume P(j) is true $\forall j \in \mathbb{Z} : 1 \le j \le k$. Now consider input k+1. Now $k+1 \ge 2$, so we return $5F(k) - 6F(k-1)$. Since $k \ge 1$, F(k) returns $3^k - 2^k$ by inductive assumption. Since $k-1 \ge 0$, and F(0) is correct by the base case, and larger values are true by inductive assumption, F(k-1) returns $3^{k-1} - 2^{k-1}$. Thus the function returns

$$5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1})$$
$$= 5 \cdot 3^k - 5 \cdot 2^k - 2 \cdot 3^k + 3 \cdot 2^k$$
$$= 3^{k+1} - 2^{k+1}$$

as desired.

Therefore, by strong induction, F(n) is correct.

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Graphs

$G = (V, E)$

Use parentheses to denote ordered set

V = set of vertices

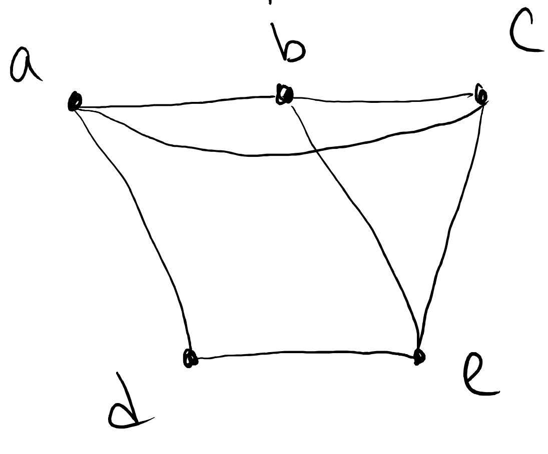
ex: $V = \{a, b, c, d, e\}$

E = set of edges

ex: $E = \{\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{d,e\}, \{b,e\}, \{c,e\}\}$

each edge is a set consisting of 2 vertex elements.

Draw this Graph:



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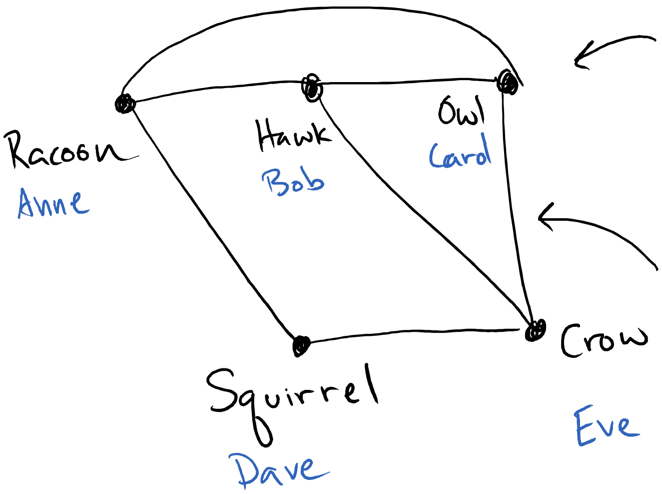
Graphs:



vertices, nodes

edges

"Niche overlap graph"

$\Downarrow$

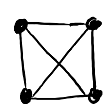
- Connection if share a food source
- Connection if friends on Facebook

Natural questions:

- Which vertex has the largest degree (degree of $v \in V$ is $|\{u : \{u,v\} \in E\}|$ "# of neighbors of u") ← most omnivorous
- Are two nodes connected? ← map
- What is the shortest path from one node to another?
- What are the fewest edges one would need to remove to separate two nodes? ← cyber attack, rail way attack

- Is the graph $G=(V,E)$ complete? (complete: $\forall u,v \in V,\ u \neq v \rightarrow \{u,v\} \in E$) "There is an edge b/t each pair of vertices"

  K_4 = complete graph on 4 vertices



- Is graph $G=(V,E)$ bipartite? [Bipartite: $\exists\, S,T \subseteq V : S \cup T = V \wedge S \cap T = \emptyset \wedge (\neg \exists \{a,b\} \in E : (a \in S \wedge b \in S) \vee (a \in T \wedge b \in T))$]