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Groups A

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Programming Assignment!

Goals

- Use de Morgan's other laws
- Prove using direct + contrapositive approaches

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Q: Which is logically equivalent to:

$\neg \exists x \in \mathbb{Z} : x^2 = x \wedge x > 1$

A: $\forall x \in \mathbb{Z},\ x^2 \neq x \vee x \leq 1$

B: $\forall x \in \mathbb{Z},\ x^2 \neq x \wedge x \leq 1$

C: $\exists x \in \mathbb{Z} :\ x^2 \neq x \vee x \leq 1$

D: $\exists x \in \mathbb{Z} :\ x^2 \neq x \vee x \leq 1$

Q: Which is logically equivalent to

$\neg \forall x \in \mathbb{Z} : 2 \mid x \vee 3 \mid x$

A: $\forall x \in \mathbb{Z},\ 2 \nmid x \vee 2 \nmid x$

B: $\forall x \in \mathbb{Z},\ 2 \nmid x \wedge 3 \nmid x$

C: $\exists x \in \mathbb{Z} :\ 2 \nmid x \vee 3 \nmid x$

D: $\exists x \in \mathbb{Z} :\ 2 \nmid x \wedge 3 \nmid x$

$2 \nmid x \equiv \neg 2 \mid x$

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Q: Which is logically equivalent to:

$\neg \exists x \in \mathbb{Z} : x^2 = x \wedge x > 1$

$\boxed{A:}$ $\forall x \in \mathbb{Z},\ x^2 \neq x \vee x \leq 1$

B: $\forall x \in \mathbb{Z},\ x^2 \neq x \wedge x \leq 1$

C: $\exists x \in \mathbb{Z} : x^2 \neq x \vee x \leq 1$

D: $\exists x \in \mathbb{Z} : x^2 \neq x \vee x \leq 1$

De Morgan's Other Rule:

$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$

$\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

- Distribute $\neg$
- Switch $\wedge \Leftrightarrow \vee$

De Morgan's Law:

$\Rightarrow \forall x \in \mathbb{Z},\ \neg(x^2 = x \wedge x > 1)$

De Morgan again

$\Rightarrow \forall x \in \mathbb{Z},\ (\neg x^2 = x \vee \neg x > 1)$

Rewrite

$\Rightarrow \forall x \in \mathbb{Z},\ (x^2 \neq x \vee x \leq 1)$

Q: Which is logically equivalent to:

$\neg \forall x \in \mathbb{Z} : 2 | x \vee 3 | x$

A: $\forall x \in \mathbb{Z},\ 2 \nmid x \vee 2 \nmid x$

B: $\forall x \in \mathbb{Z},\ 2 \nmid x \wedge 3 \nmid x$

C: $\exists x \in \mathbb{Z} : 2 \nmid x \vee 3 \nmid x$

$\boxed{D:}$ $\exists x \in \mathbb{Z} : 2 \nmid x \wedge 3 \nmid x$

De Morgan's Law

$\Rightarrow \exists x \in \mathbb{Z} : \neg(2 | x \vee 3 | x)$

De Morgan Again

$\Rightarrow \exists x \in \mathbb{Z} : (2 \nmid x \wedge 3 \nmid x)$

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Direct Proof

Use: Prove $P \to Q$

Structure:

Assume P. Explain, explain, explain. Therefore Q.

Use: Prove $\forall x \in S,\ P(x) \to Q(x)$

Structure

Let $x \in S$, and assume $P(x)$.

Explain, explain, explain.

Therefore $Q(x)$

$\boxed{\text{If domain of } x \text{ is obvious, can leave off "Let } x \in S\text{," like proofs on worksheet. But it is clearer if write it. So worksheet proof should have "Let } a, b \in \mathbb{Z}\text{".}}$

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You've already written a direct proof!

Inductive step: $\forall k \in \mathbb{Z}, \left(k \geq \text{base case} \wedge P(k) \right) \rightarrow P(k+1)$

Let $k \geq$ [base case]. Assume for induction $P(k)$ is true.

Explain
Explain
Explain

Therefore, $P(k+1)$ is true

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Q: Prove: If $a|b$ and $b|c$ then $a|c$.

$(a|b \equiv \exists e \in \mathbb{Z} : ae = b)$

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Q: Prove: If $a|b$ and $b|c$ then $a|c$.
$(a|b \equiv \exists e \in \mathbb{Z} : ae = b)$

Let $a, b, c \in \mathbb{Z}$. Assume $a|b$ and $b|c$. This means $\exists\ e, f \in \mathbb{Z}$ such that $b = ae$ and $c = bf$. Then

$$c = bf = (ae)f = a(ef).$$

That means $a|c$, since $c = ak$, for $k = ef$, an integer.

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Proof By Contrapositive

- Use: Prove $P \to Q$

P	Q	$\neg Q$	$\neg P$	$P \to Q$	$\neg Q \to \neg P$
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

$$P \to Q \equiv \neg Q \to \neg P$$

Structure:

We prove the contrapositive. Assume $\neg Q$. Explain explain, explain. Therefore, $\neg P$.

- Use: Prove $\forall x \in S,\ P(x) \to Q(x)$

Structure

We prove the contrapositive
Let $x \in S$ and assume $\neg Q(x)$

Explain, explain, explain.

Therefore, $\neg P(x)$.

Q: If a^2 is not divisible by 4, then a is odd.

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Q: If a^2 is not divisible by 4, then a is odd.

We prove the contrapositive. Let $a \in \mathbb{Z}$. Suppose a is even. Then $\exists k \in \mathbb{Z} : 2k = a$. This means $a^2 = 4k^2$. Since k^2 is an integer, $4 | a^2$.

* There is an implied "for all" in the proof statement. (otherwise it is a predicate & we can't prove true or false). Therefore, we need "Let $a \in \mathbb{Z}$."

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Iff Proofs

Iff = "if and only if.

Use: $P \leftrightarrow Q$

$$(P \to Q) \land (Q \to P) \equiv P \leftrightarrow Q$$

(use truth table proof)

Structure

¶ For the forward direction, [Proof of $P \to Q$]

¶ For the backwards direction, [Proof of $Q \to P$]

Could be direct or contrapositive