CS200 - Problem Set 3

1. **[2 point each]** Simplify each of the following expressions, where p denotes a statement, and T and F are the Boolean constants *true* and *false*. Hint: each answer is one of p, T, or F. No proof needed, no steps need be shown.

   (a) $T \wedge p$

   (b) $F \wedge p$

   (c) $T \vee p$

   (d) $F \vee p$

   (e) $T \to p$

   (f) $p \to p$

   (g) $p \vee \neg p$

2. Consider the following statement:

$$\forall x, \exists y : (y > x) \wedge (\forall z, ((z \neq y) \wedge (z > x)) \to (z > y))$$

   (a) **[6 points]** If the domain of x, y and z is $\mathbb{Z}$, state whether the statement is true or false. Justify your answer.

   (b) **[6 points]** If the domain of x, y and z is $\mathbb{R}$, state whether the statement is true or false. Justify your answer.

3. **[6 points each]** Let S be a set of students in a class and $f(s)$ be the score obtained by student s in an exam. Translate the English description of each of the following predicates or statements into a logical formula using only mathematical notation. To get full credit, your answer should only use mathematical notation, and your response should be approximately as concise as mine. Use the $\equiv$ symbol in your answer. For example, for part a, you should write $H(n) \equiv \ldots$.

   These are challenging! To start each one, think about whether it should start with a "for all," or "there exists" quantifier. Next think about what set the domain should be for that quantifier. Then keep going :)

   Things to remember: You can use previously defined predicates in later parts. For example, you will probably use "$H(n)$" in almost all later parts. Make sure you don't mix types: don't put a number as input to a function whose input should be a student, or put a $\wedge$ between numbers ($\wedge$ can only go between predicates or statements)! Finally, make sure you quantify any new variables you create, and don't quantify the input to the predicate you are trying to translate.

   (a) Predicate $H(n)$ asserts: n is the highest score that any student got on the exam.

(b) Predicate $B(s)$ asserts: student s got the highest score. (This one doesn't need a quantifier)

(c) Statement p asserts: at least two students got the highest score.

(d) Predicate $M(n)$ asserts: if any two students got the same score, that score is at least n.

(e) Predicate $R(s)$ asserts: student s got 10 points less than the highest score.

(f) Statement t asserts: the second highest score in the class is 10 points less than the highest score.

4. You will likely find the following (from *Discrete Mathematics, an Open Introduction* by Levin; click here to go to online version) helpful for this problem.

Set Theory Notation

$\{,\}$	We use these **braces** to enclose the elements of a set. So $\{1, 2, 3\}$ is the set containing 1, 2, and 3. (Roster notation)
$:$	$\{x : x > 2\}$ is the set of all x **such that** x is greater than 2. (set-builder notation)
$\in$	$2 \in \{1, 2, 3\}$ asserts that 2 is **an element of** the set $\{1, 2, 3\}$.
$\notin$	$4 \notin \{1, 2, 3\}$ because 4 **is not an element of** the set $\{1, 2, 3\}$.
$\subseteq$	$A \subseteq B$ asserts that A **is a subset of** B: every element of A is also an element of B.
$\subset$	$A \subset B$ asserts that A **is a proper subset of** B: every element of A is also an element of B, but $A \neq B$.
$\cap$	$A \cap B$ is the **intersection of** A **and** B: the set containing all elements which are elements of both A and B.
$\cup$	$A \cup B$ is the **union of** A **and** B: is the set containing all elements which are elements of A or B or both.
$\times$	$A \times B$ is the **Cartesian product of** A **and** B: the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$.
$\setminus$	$A \setminus B$ is A **set-minus** B: the set containing all elements of A which are not elements of B.
$\overline{A}$	The **complement of** A is the set of everything which is not an element of A. (Depends on what "everything" is. Define U = universal set to be everything.)
$\|A\|$	The **cardinality (or size) of** A is the number of elements in A.

(a) **[2 points each]** Describe the following sets in roster notation (list the first few elements). If the set is also "famous" give its symbol.

i. $A = \{2^x : x \in \mathbb{N}\}$

ii. $B = \{x : x \text{ is even and } x \in \{1, 3, 5\}\}$

iii. $C = \{x \geq 0 : x \text{ is even or } x \text{ is odd}\}$

(b) **[6 points each]** Write the following in set-builder notation using as concise and mathematical notation as possible

i. $\{2, 4, 6, 8, 10, 12\}$

ii. $\{1, 3, 5, 7, 9, 11, \dots\}$

iii. $\{1, 4, 9, 16, 25, 36, \dots\} \cap \{2, 4, 6, 8, 10, \dots\}$

iv. $\{a, e, i, o, u\}$

(c) **[2 points each]** Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$

i. What is $A \times B$?

ii. What is $|A \times B|$?

iii. Is $A \subset B$?

iv. Is $A \subseteq B$?

v. Is $A \subset A$?

vi. What is $A \setminus B$? (I will also use the notation $A - B$ to mean $A \setminus B$.)

vii. What is $A \cup B$?

viii. What is $A \cap B$?

(d) **[2 points]** Which of the following are the empty set:

i. $\{x : x \text{ is odd and } 7 < x < 9\}$

ii. $\{0\}$

iii. $\{\emptyset\}$

iv. $\mathbb{Z} \cap \mathbb{Q}$

(e) **[6 points]** Let A and B be sets with $|A| = |B|$ such that $|A \cup B| = 7$ and $|A \cap B| = 3$. What is $|A|$? Explain.

(f) **[6 points]** Find sets A and B such that $A \subset B$ and $A \in B$.

(g) **[6 points]** Does the empty set contain itself?

5. How long did you spend on this homework?