

CS200 - Problem Set 1

Due: Monday, Feb. 18 to Canvas by 9pm

Please read the sections of the syllabus on problem sets and honor code before starting this homework.

1. Explain what is wrong with each of the following proofs:

(a) [2 points] [DM 2.5.14](#) (Click on the link in a pdf to go to the textbook section 2.5, then go to problem 14)

(b) [6 points]

Proof:

Let $P(n)$ be the predicate that you can make n cents of postage using 3-cent and 7-cent stamps. We will prove using induction that $P(n)$ is true for all $n \geq 14$.

Base case: When $n = 14$, note that you can make 14 cents using two 7-cent stamps.

Inductive case: Let $k \geq 14$. Suppose $P(k)$ is true. Then there are positive numbers x (corresponding to the number of 3-cent stamps) and y (corresponding to the number of 7-cent stamps) such that

$$k = 3x + 7y. \tag{1}$$

Now we can remove two 7-cent stamps and add five 3-cent stamps. Then we have

$$3(x + 5) + 7(y - 2) = 3x + 15 + 7y - 14 = k + 1. \tag{2}$$

Thus we can create $k + 1$ -cents worth of postage using 7-cent stamps and 3-cent stamps.

Therefore, by induction $P(n)$ is true for all $n \geq 14$.

(c) [6 points]

Proof:

Let $P(n)$ be the predicate that the sum of the first n odd numbers is n^2 . We will prove $P(n)$ is true for all $n \geq 1$.

Base case: When $n = 1$, the sum of the first odd number is $1 = 1^2$, so $P(1)$ is true.

Inductive step: For $k = 2$, we look at $1 + 3$, which equals 2^2 . For $k = 3$, we look at $1 + 3 + 5$, which equals 3^2 . Continuing in this way, we see that sum of the first k odd numbers is k^2 .

Thus for all $n \geq 1$, $P(n)$ is true.

(d) [6 points]

Proof:

Let $P(n)$ be the predicate that the sum of the first n numbers is $(n^2 + n)/2$. We will prove $P(n)$ is true for all $n \geq 1$.

Base case: When $n = 1$, the sum of the first number is 1, which equals $(1^2 + 1)/2$.

Inductive step: Assume $P(k)$ is true for $k \geq 1$. This means

$$1 + 2 + \cdots + k = (k^2 + k)/2. \quad (3)$$

Now if we add $k + 1$ to both sides, we have

$$P(k) + (k + 1) = (k^2 + k)/2 + (k + 1). \quad (4)$$

Doing some math, we find that the right hand side is equal to $((k+1)^2 + (k+1))/2$, so $P(k + 1)$ is true.

Thus for all $n \geq 1$, $P(n)$ is true.

(e) [6 points]

Proof:

Let $P(n)$ be the predicate that the sum of the first n numbers is $(n^2 + n)/2$. We will prove $P(n)$ is true for all $n \geq 1$.

Base case: When $n = 1$, the sum of the first number is 1, which equals $(1^2 + 1)/2$.

Inductive step: Assume $P(k)$ is true for $k \geq 1$. Then if we consider $P(k + 1)$, we have

$$1 + 2 + \cdots + k + 1 = ((k + 1)^2 + (k + 1))/2. \quad (5)$$

Now if we subtract $k + 1$ from both sides and do some algebra, we have

$$1 + 2 + \cdots + k = (k^2 + k)/2, \quad (6)$$

which is $P(k)$, which we know is true.

Thus for all $n \geq 1$, $P(n)$ is true.

(f) [6 points] (Challenge!) Explain what is wrong with the following inductive proof that all Middlebury students have the same eye color. I find it easiest to describe the issue by using the “ladder” analogy from class.

Proof: Let $P(n)$ be the predicate that any group of n Middlebury students have the same eye color. We will prove $P(n)$ is true for all $n \geq 1$.

Base case: $P(1)$ is true because any one Middlebury student has the same eye color as themselves.

Inductive case: Let $k \geq 1$. Assume for induction that any set of k Middlebury students have the same eye color. Now let’s consider any set of $k + 1$ Middlebury

students. If we look at the first k of those $k + 1$ students, by our inductive assumption they must all have the same eye color. However, if we look at the last k of those $k + 1$ students, by our inductive assumption, they must also all have the same eye color. Now the second student must be part of the first set of k and the last set of k , so all $k + 1$ students must have the same eye color as this second student. Thus, any set of $k + 1$ Middlebury students have the same eye color, so $P(k + 1)$ is true.

Therefore, by induction, $P(n)$ is true for all $n \geq 1$.

- (g) [3 points] Why is it important to be able to find errors in inductive proofs?
2. We can use induction to prove a recursive algorithm is correct. However, it involves a couple of stylistic ingredients that are a little different from a mathematical proof. I will prove `Mult` is correct for you, and highlight some important parts. Then you should prove `Log` is correct.

Algorithm 1: `Mult`(m)

```

Input  : Non-negative integer  $m$ 
Output:  $m^2$ 
/* Base Case
1 if  $m == 0$  then
2 |   return 0;
3 else
4 |   // Recursive step
   |   return  $2 \times m - 1 + \text{Mult}(m - 1)$ ;
5 end

```

Proof: Let $P(n)$ be the predicate that `Mult`(n) correctly outputs n^2 . We will prove using induction that $P(n)$ is true for all $n \geq 0$. [NOTE: n is the variables we will use for induction, while m is the variable we will use when referencing the pseudocode. You can use the same variables, but it can get confusing in more complicated proofs.]

For the base case, let $n = 0$, so we need to analyze `Mult`(0). In this case, the `If` statement is true at line 1, and the algorithm returns 0. Since $0 = 0^2$, the output is correct. [NOTE: I've analyzed what happens in the code in the base case, and compared to what I wanted the output to be. I reference line numbers to make it clear what I'm talking about.]

For the inductive step, let $k \geq 0$ and assume $P(k)$ is true. Now let's consider what happens when we run `Mult`($k+1$). Since $k \geq 0$, then $k + 1 \geq 1$, so the algorithm does not trigger the base case but goes to the `else` case in line 4. [NOTE: You need to explain why the algorithm does not trigger the base case in the code] and returns $2(k + 1) - 1 + \text{Mult}((k + 1) - 1) = 2(k + 1) - 1 + \text{Mult}(k)$. By inductive assumption, `Mult`(k) returns k^2 . [NOTE: This is the critical inductive step! You need to replace the output of the recursive call with whatever you can assume the output of the function should be by inductive assumption. Then once you have done this replacement, you

normally need to do some math, as follows:] Using this assumption, we have that at line 4 the algorithm returns

$$2(k + 1) - 1 + k^2 = k^2 + 2k + 1 = (k + 1)^2. \quad (7)$$

This is the correct output for input $m = k + 1$, so $P(k + 1)$ is true.

Therefore, by induction, $P(n)$ is true for all $n \geq 0$.

[11 points] Prove the following algorithm evaluates the log (base 2) of the input. Depending on your background, you may need to learn or review logarithms and exponentials. [Khan Academy](#) has some useful videos, or there are many other online resources that have basic info on logarithms and exponentials. Hint: your induction variable should correspond to g .

Algorithm 2: $\text{Log}(m)$

Input : Positive integer m of the form 2^g

Output: g

/* Base Case

1 **if** $m == 1$ **then**

2 | return 0;

3 **else**

4 | // Recursive step

4 | return $1 + \text{Log}(m/2)$;

5 **end**

*/

3. How long did you spend on this homework?