S.KIMMEL

Goals

- Write Proof By Contradiction
- Write Strong inductive proof.

Test Info

- All topics up to Iff proofs on 3/6
- All PSets up to 4
- Monday : review . Topics → Canvas Discussion until Saturday
- Wed @ ~~noon~~ 10am → Frid @ 6 pm
- Reserve 3 hours
- 1 page, single sided hand written notes "cheat sheet"

Quiz on Canvas

Short PSet 5

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Goals

- Describe & write proof by contradiction
- Describe & write proof by strong induction

Proof by Contradiction

Use: any statement P

Proof needs to do two things

$$\left.\begin{array}{ll} \text{① (Direct)} & \neg P \rightarrow Q \\ \text{② (Direct)} & \neg P \rightarrow \neg Q \\ \hline & \therefore P \end{array}\right\} \text{Most common}$$

Structure

For contradiction assume $\neg P$

Explain explain... Q

Explain explain... $\neg Q$, a contradiction.

Thus, P is true

When start, don't know what Q is... you need to keep your eye out for what might be the contradiction.

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Q: Prove $\sqrt{2}$ is irrational

* Not of form $P \to Q$

A: For contradiction, suppose $\sqrt{2}$ is rational. Then $\exists\, a, b \in \mathbb{Z} : \frac{a}{b} = \sqrt{2}$ where the fraction is fully simplifed, so $\neg \exists\, c \in \mathbb{Z} : c|a \wedge c|b$.
Squaring both sides, we have

$$a^2 = 2b^2.$$

Thus $2|a^2$. But we've previous proved this implies $2|a$. This means $\exists\, m \in \mathbb{Z} : 2m = a$. Plugging in, we have

$$4a^2 = 2b^2.$$

Dividing by 2, we get

$$2a^2 = b^2.$$

But this means $2|b^2$, and so $2|b$. But this means $2|a$ and $2|b$, which contradicts the fact that $\frac{a}{b}$ is fully simplifed. $\square$

Q: Prove: $\neg \exists\, x, y \in \mathbb{Z} : x^2 = 4 \quad +2$

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Try: $\neg \exists x, y \in \mathbb{Z}: x^2 = 4y + 2$ (1st step: is x^2 even or odd)

Suppose for contradiction, there exist $x, y \in \mathbb{Z}: x^2 = 4y+2$. Then x^2 is even, so x is even. Thus $\exists m \in \mathbb{Z}: x = 2m$. Plugging in, and solving for y, we have

$$y = \frac{4m^2 - 2}{4} = m^2 - \frac{1}{2}$$

Since m is an integer, $m^2 - \frac{1}{2}$ is <u>not</u> an integer, a contradiction.

Other ways to get contradictions:

To Prove P $\Longrightarrow$ Prove $\neg P \to P$

To Prove $P \to Q$ $\Longrightarrow$ Prove $(P \wedge \neg Q) \to Q$

$\neg (P \to Q) \equiv P \wedge \neg Q$

To Prove P $\Longrightarrow$ $\neg P \to$ something obviously false

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Strong Induction Proof Structure

Set-up

Let $P(n)$ be the predicate ________________ We will prove $P(n)$ is true for all $n \geq$ b.c.

(↑ base case)

Base-case

Base-case: We prove $P(\text{b.c.})$ is true. ... (Sometimes multiple b.c. $P(0)$, $P(1)$)

Inductive case

Inductive case: Assume for strong induction that $P(j)$ is true for all j such that $\text{b.c.} \leq j \leq k$.

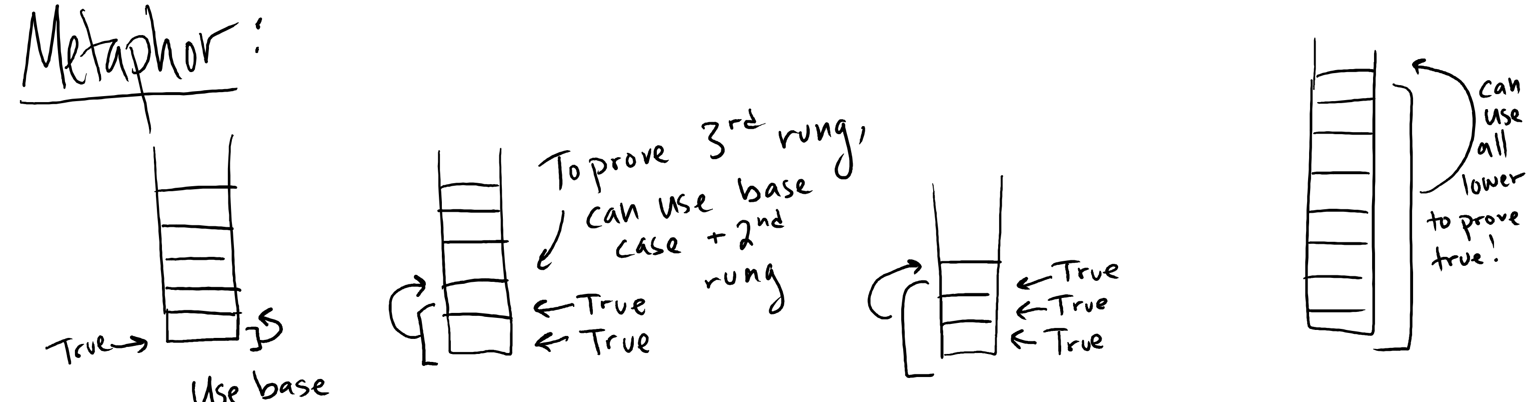
⋮

Therefore $P(k+1)$ is true

Conclusion

By strong induction, we conclude $P(n)$ is true for all $n \geq$ b.c.

Metaphor:



Usually: just need a couple of lower rungs, but we assume all lower rungs are true to make things simpler