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Counting Infinite Sets

Given infinite time, a computer can do infinitely many thing... but can it do everything? ... will answer in 301. Today, tools.

$$\left.\begin{array}{l} |\mathbb{N}| = \infty \\ |\underbrace{\{x > 0 : x \text{ is even}\}}_{E}| = \infty \end{array}\right\} \text{Same infinity?}$$

Q: $|N| = |E|$?

A) $|N| < |E|$ B) $|N| = |E|$ C) $|N| > |E|$

$\Uparrow$

For sets A, B, $|A| = |B| \iff \exists\ f : A \to B$ where f is bijective.

eg. $f(x) = 2x$, $f : N \to E$

Another way to see bijection:

$$\begin{array}{c} \overbrace{1 \quad\ 2 \quad\ 3}^{A} \\ \updownarrow \quad \updownarrow \quad \updownarrow \quad \cdots \\ \underbrace{2 \quad\ 4 \quad\ 6}_{B} \end{array}$$

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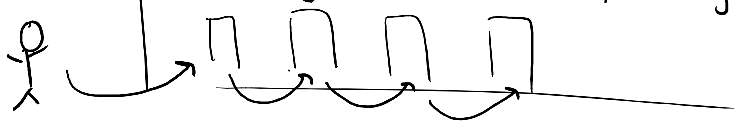
Another way to see infinities are same

Hotel w/infinite rooms, all occupied

New guy

Everyone moves down, new guy gets room, all rooms occupied

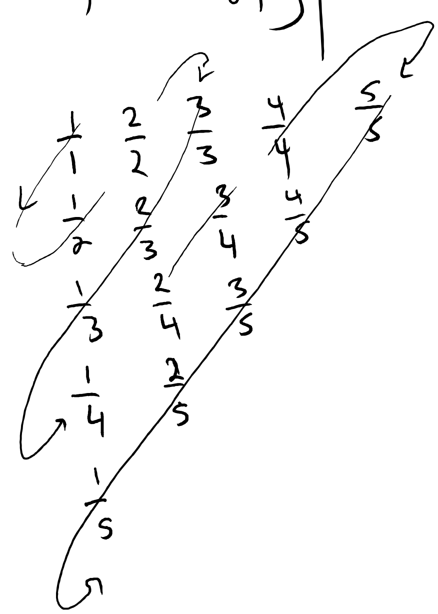


$|\mathbb{N}| = |\mathbb{N}+1|$

Q: Show $|\mathbb{N}| = |\{x \in \mathbb{Z} : x > 10\}|$ $\qquad f(x) = x + 10$

Show $|\mathbb{N}| = |\mathbb{Z}|$ $\qquad f(x) = \begin{cases} \frac{x}{2}, & x \text{ even} \\ -\left(\frac{x+1}{2}\right) & x \text{ odd,} \end{cases}$

Show $|\mathbb{N}| = \left|\left\{ \frac{x}{y} : x, y \in \mathbb{Z}, |x| < |y| \right\}\right|$



We call any set that has the same size as $\mathbb{N}$, <u>Countably infinite</u>

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Fact: $|\mathbb{N}| < \left|\{x \in \mathbb{R} : 0 < x < 1\}\right|$ ← $\mathbb{R}_1$

We call sets that are larger than $\mathbb{N}$ <u>uncountably infinite</u>

Proof uses Diagonalization:

1. QUIET
2. STONE
3. OFFER
4. CLEAR
5. PHONE

New Word not on list:

S	L	E	E	T
↑	↑	↑	↑	↑
Not Q	Not T	NoT F	NOT A	NoT E

Pf: Suppose for contradiction there is a bijection from $\mathbb{N}$ to $\mathbb{R}_1$

$1 \longleftrightarrow 0.d_{11}d_{12}d_{13}\ldots$

$2 \longleftrightarrow 0.d_{21}d_{22}d_{23}\ldots$

$3 \longleftrightarrow 0.d_{31}d_{32}d_{33}\ldots$ ← continue to infinity

d_{ij} = j^{th} decimal place of i^{th} real #

$\frac{1}{2} = 0.500000\ldots$

Now consider the number

$d = 0.d_1 d_2 d_3 d_4 \ldots$

$$d_k = \begin{cases} 4 & \text{if } d_{kk} \neq 4 \\ 5 & \text{if } d_{kk} = 4 \end{cases}$$

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$1 \leftrightarrow 0.15434\ldots$
$2 \leftrightarrow 0.41198\ldots \quad \Rightarrow \quad 0.445\ldots$
$3 \leftrightarrow 0.97422\ldots$

d is a number in $\mathbb{R}_1$, but d is not in the list! But we claimed all numbers in $\mathbb{R}_1$ are on the list $\rightarrow$ contradiction

If $|A| = |\mathbb{N}|$, we call A countably infinite
If $|A| > |\mathbb{N}|$, we call A uncountably infinite

Q: Let $F: \{f : \mathbb{N} \rightarrow \{0,1,2,3,4,5,6,7,8,9,10\}\}$

Show $|F| > |\mathbb{N}|$

Pf: Suppose for contradiction there is a bijection between F and $\mathbb{N}$

$1 \leftrightarrow f_1 = f_1(1), f_1(2), f_1(3) \ldots$
$2 \leftrightarrow f_2 = f_2(1), f_2(2), f_2(3) \ldots$
$\vdots$

Can create a new function that differs at every position