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Goals

Calculate expectation values using

- Indicator random variable
- Linearity of expectation

Other

- Pseudo Code
- Summation tricks : $\sum(a+b) = \sum a + \sum b$
  $\sum Aa = A \cdot \sum a$

<u>def</u>: Given a sample space S, a random variable X is a function $X: S \to \mathbb{R}$.

ex: Let S be the sample space consisting of all possible outcomes of 4 coin tosses. Let X be the number of heads that occur.

Q: What is $X(T,H,H,H)$? What is $X(T,T,H,T)$?

A: 1, 3 B: 2, 2 $\boxed{\text{C: } 3, 1}$ D: 4, 4

<u>def</u>: The <u>expected</u> or <u>average</u> value of a random variable X is

$$\mathbb{E}[X] = \sum_{i \in S} \Pr(i)\, X(i).$$

Q: From previous example, what is $\mathbb{E}[X]$? (The averge number of heads in 4 coin flips.)

A) 1 $\boxed{\text{B) } 2}$ C) 2.5 D) 4

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- I'm guessing you didn't do the following:

$$\mathbb{E}[X] = \sum_{i \in S} \Pr(i) X(i)$$ (2^4 elements of sample space!)

$$\mathbb{E}[X] = \sum_{\substack{i \in S: \\ X(i)=0}} \Pr(i) \cdot 0 + \sum_{\substack{i \in S \\ X(i)=1}} \Pr(i) + \sum_{\substack{i \in S \\ X(i)=2}} \Pr(i) \cdot 2$$

$$+ \sum_{\substack{i \in S \\ X(i)=3}} \Pr(i) 3 + \sum_{\substack{i \in S \\ X(i)=4}} \Pr(i) 4$$

... good practice to finish on your own!

$\Pr(i) = \frac{1}{2^4} = \frac{1}{16}$ in all cases.

$|\{i \in S : X(i) = 0\}| = 1$

$|\{i \in S : X(i) = 1\}| = \binom{4}{1} = 4$

$|\{i \in S : X(i) = 2\}| = \binom{4}{2} = 6$

$|\{i \in S : X(i) = 3\}| = \binom{4}{3} = 4$

$|\{i \in S : X(i) = 4\}| = 1$

$$\mathbb{E}[X] = \frac{1}{16}\left(1 \cdot 0 + 4 \cdot 1 + 6 \cdot 2 + 4 \cdot 3 + 1 \cdot 4\right)$$
$$= \frac{1}{16}(32) = 2$$

- Instead you used indicator random variables + linearity of expectation (without knowing!)

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Indicator Random Variable:

def: An indicator random variable X is a random variable such that $X: S \to \{0,1\}$.

An indicator random variable is associated with an event $E \subseteq S$

$$E = \{i \in S : X(i) = 1\}$$

Normally write as X_E where

$$X_E(s) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathbb{E}[X_E] = \sum_{i \in S} \Pr(i)\, X_E(i)$$

$$= \sum_{\substack{i \in S: \\ X_E(i)=0}} \Pr(i) \cdot 0 + \sum_{\substack{i \in S \\ X_E(i)=1}} \Pr(i)$$

$$= \sum_{i \in E} \Pr(i) = \Pr(E)$$

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Linearity of Expectation

Let $Y_1, Y_2, \ldots, Y_n$ be random variables on a sample space S. Let $a_1, a_2, \ldots a_n \in \mathbb{R}$. Let Y be a random variable s.t.

$$Y = \sum_{k=1}^{n} a_k \cdot Y_k \qquad \left(\text{That is } \forall i \in S,\ Y(i) = \sum_{k=1}^{n} a_k \cdot Y_k(s)\right)$$

Then
$$\mathbb{E}[Y] = \sum_{k=1}^{n} a_k \, \mathbb{E}[Y_k]$$

Ex: Let X_k be the indicator random variable that takes value 1 if k^{th} coin flip is Heads.

$$X = \#\text{ of heads in 4 coin tosses}$$

Then $\quad X = \sum_{k=1}^{4} X_k$

e.g.
$$X(H,T,T,H) = \overset{1}{X_1(H,T,T,H)} + \overset{0}{X_2(H,T,T,H)} + \underset{0}{X_3(H,T,T,H)} + \underset{1}{X_4(H,T,T,H)} = 2$$

$$\mathbb{E}[X] = \sum_{k=1}^{4} \mathbb{E}[X_k] = \sum_{k=1}^{4} \Pr($$

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Q: What is average number of heads in 4 coin flips

1. See <u>average</u>. Need to figure out sample space, random variable

$\{H, T\}^4$ $X(i) = \#$ heads

2. Write X as a weighted sum of indicator random variables (can do in several steps)

a) Think about what events causes X to increase

Event: 1^{st} flip is H $\rightarrow$ increases X by 1
Event: 2^{nd} flip is H $\rightarrow$ "
" 3^{rd} "
" 4^{th}

b) Create indicator random variables for each event & write X as sum, based on how much each increases

$$X = 1 \cdot X_1 + 1 \cdot X_2 + 1 \cdot X_3 + 1 \cdot X_4 = \sum_{k=1}^{4} X_k$$

3. Use linearity of expectation:

$$\mathbb{E}[X] = \sum_{k=1}^{4} \mathbb{E}[X_k]$$

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5. Use $\mathbb{E}[X_E] = \Pr(E)$ for indicator random variables.

$$\mathbb{E}[X] = \sum_{k=1}^{4} \Pr(k^{th} \text{ flip is heads}) = \sum_{k=1}^{4} \frac{1}{2} = 2$$

Super powerful!

ex: Time complexity of random binary search is $O(\log n)$