Name ____________________

CS200 - Midterm 1 Review Questions

1. Let S be the set of all people who have every lived. Let $G(x, y)$ be the predicate, x is the grandmother of y, for $x, y \in S$. Let $C(x, y)$ be the predicate x and y are cousins for $x, y \in S$. Write the following statements and predicates using math:

   (a) "All people have at least two grandmothers."

   (b) "Every pair of cousins share a grandmother."

   (c) "None of Person x's cousins are grandmothers "

   (d) (Challenge) "All of Person x's children except one are childless."

   Solution

   (a) $\forall x \in S, \exists y, z \in S : (y \neq z) \wedge G(y, x) \wedge G(z, x)$.

   (b) $\forall x, y \in S, C(x, y) \rightarrow (\exists z \in S : G(z, x) \wedge G(z, y))$.

   (c) $\forall y \in S, C(x, y) \rightarrow (\neg\exists w \in S : G(y, w))$.

   (d) $(\exists y \in S : G(x, y)) \wedge (\neg\exists w \in S : C(w, y))$.

2. You meet a group of 50 orcs. You know orcs are either honest or corrupt. Suppose you know that at least one of the orcs is honest. You also know that given any two of the orcs, at least one is corrupt. Let G be the set of orcs, and $D(g)$ is the predicate "orc g is corrupt."

   (a) How many of the orcs are corrupt and how many are honest?

   (b) Express "At least one orc is honest" using math.

   (c) Express "Given any two orcs, at least one is corrupt" using math

   Solution

   (a) There is one honest orc. If there were two honest orcs then you could have a pair where both are honest, and we know that any pair has to have a corrupt orc.

   (b) $\exists g \in G : \neg D(g)$.

   (c) $\forall x, y \in G, x \neq y \rightarrow (D(x) \vee D(y))$.

3. Consider ways of proving: $14|a$ if and only if $2|a$ and $7|a$.

   (a) How would you start a direct proof of the forward direction?

   (b) How would you start a direct proof of the backwards direction?

(c) How would you start a contrapositive proof of the forward direction?

(d) How would you start a contrapositive proof of the backwards direction?

(e) How would you start a proof by contradiction of the forward direction?

(f) How would you start a proof by contradiction of the backwards direction?

(g) In this case it turns out that a direct proof is the best approach in both cases. If you'd like practice, complete the proof. Also, explain why the other approaches are less good.

Solution

(a) For the forward direction, assume $14|a$. This means $\exists m \in \mathbb{Z} : 14 \times m = a$. But that means $2 \times (7 \times m) = a$, so $2|a$ and $7 \times (2 \times m) = a$, so $7|a$.

(b) For the backwards direction, assume $2|a$ and $7|a$. That means $\exists m, n \in \mathbb{Z} : 2m = a \wedge 7n = a$. Thus $2m = 7n$. Since 2 divides the right hand side, 2 must divide the right hand side so $7n$ is even. However, since 7 is odd, in order for $7n$ to be even, n must be even, which means $2|n$. Then $\exists g \in \mathbb{Z} : 2g = n$. Plugging in, we have $7 \times 2g = a$, so $14|a$.

(c) For the forward direction, assume $\neg 2|a$ or $\neg 7|a$. [This approach is hard because saying something is not divisible by something doesn't give you much to work with.]

(d) For the backwards direction, assume $\neg 14|a$. [This approach is hard because saying something is not divisible by something doesn't give you much to work with.]

(e) For the forward direction, assume $14|a$ but $\neg 2|a$ or $\neg 7|a$. [It is not really that helpful to additionally assume $\neg 2|a$ or $\neg 7|a$ (see comment above), and just assuming $14|a$ is already enough to easily prove the result, as in the first part.]

(f) For the backwards direction, assume $2|a$ and $7|a$ but $\neg 14|a$. [Again it is not really that helpful to additionally assume $\neg 14|a$ (see comment above), and just assuming $2|a$ and $7|a$ is already enough to easily prove the result, as in the first part.]

4. **[11 points]** Prove using induction that program `ProductThing`(n) returns a number less than or equal to n^n for all $n \geq 1$.

Algorithm 1: `ProductThing`(m)

Input : $m \in \mathbb{Z}$ such that $m \geq 1$
Output: Something.

```
// Base Case
if m == 1 then
    return 1;
else
    // Recursive step
    return ProductThing(m − 1) × m
end
```

Solution Let $P(n)$ be the predicate that $\texttt{ProductThing}(n)$ returns a number less than or equal to n^n. We will prove using induction that $P(n)$ is true for all $n \geq 1$.

Base case: $P(1)$ is true because when the input to the algorith is 1, the algorithm returns 1 at line 2, which is equal to 1^1.

Inductive step: Let $k \geq 1$. Assume for induction that $P(k)$ is true. Let's consider what happens when the input to $\texttt{ProductThing}$ is $k+1$. Then since $k+1>1$, we go to Line 4 and return $\texttt{ProductThing}(k+1-1) \times (k+1) = \texttt{ProductThing}(k) \times (k+1)$. From our inducttive assumption, we know $\texttt{ProductThing}(k) \leq k^k$. Multiplying both sides of this inequality by $(k+1)$, we have

$$\texttt{ProductThing}(k) \times (k+1) \leq k^k \times (k+1). \tag{1}$$

Now since $k \geq 1$, we know $k^k \leq (k+1)^k$. Multiplying both sides of this inequality by $(k+1)$ we have $k^k(k+1) \leq (k+1)^k \times (k+1) = (k+1)^{k+1}$. Thus (using the transitive property of inequalities)

$$\texttt{ProductThing}(k) \times (k+1) \leq (k+1)^{k+1}. \tag{2}$$

Therefore $P(k+1)$ is true.

Thus, $P(n)$ is true for all $n \geq 1$.

5. **Sorry I introduce a typo! This is not actually true!!** Prove that if $a, b \in \mathbb{Z}$, then $a^2 - 2b - 2 \neq 0$ using a proof by contradiction (BOP 6A6)