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Counting Infinite Sets

Given infinite time, a computer can do infinitely many thing...
but can it do everything? ... will answer in 301. Today, tools.

$$\left.\begin{array}{l} |\mathbb{N}| = \infty \\ |\underbrace{\{x>0 : x \text{ is even}\}}_{E}| = \infty \end{array}\right\} \text{Same infinity?}$$

Q: $|N| = |E|$?

A) $|N| < |E|$ B) $|N| = |E|$ C) $|N| > |E|$

For sets A, B, $|A| = |B| \iff \exists\, f: A \to B$ where f is bijective (surjective & injective)

$\boxed{\text{To prove } |\mathbb{N}| = |E|, \text{ use proof by example}}$, Create bijective function: $f(x) = 2x$, $f: \mathbb{N} \to E$

Another way to see bijection:

$$\begin{array}{c} \overbrace{\begin{array}{cccc} 1 & 2 & 3 & \\ \updownarrow & \updownarrow & \updownarrow & \cdots \\ 2 & 4 & 6 & \end{array}}^{A} \\ \underbrace{\hphantom{1 \quad 2 \quad 3 \quad \cdots}}_{B} \end{array}$$

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Q: Show $|\mathbb{N}| = |\{x \in \mathbb{Z} : x > 10\}|$

Show $|\mathbb{N}| = |\mathbb{Z}|$

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Q: Show $|\mathbb{N}| = |\{x \in \mathbb{Z} : x > 10\}|$

$f(x) = x + 10$

$f: \mathbb{N} \to \{x \in \mathbb{Z} : x > 10\}$

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 11 & 12 & 13 & 14 & 15 & \cdots \end{array}$$

Show $|\mathbb{N}| = |\mathbb{Z}|$

$$f(x) = \begin{cases} \frac{(x-1)}{2} & x \text{ odd} \\ \frac{-x}{2} & x \text{ even} \end{cases}$$

$f: \mathbb{N} \to \mathbb{Z}$

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & -1 & 1 & -2 & 2 & \cdots \end{array}$$

We call any set that has the same size as $\mathbb{N}$, <u>Countably infinite</u>

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Fact: $|\mathbb{N}| < |\{x \in \mathbb{R} : 0 < x < 1\}|$ ← $\mathbb{R}_1$

We call sets that are larger than $\mathbb{N}$ <u>uncountably infinite</u>

Proof uses Diagonalization:

1. QUIET
2. STONE
3. OFFER
4. CLEAR
5. PHONE

New Word not on list:

S	L	E	E	T
↑	↑	↑	↑	↑
Not Q	Not T	NoT F	NOT A	NoT E

Pf: Suppose for contradiction there is a bijection from $\mathbb{N}$ to $\mathbb{R}_1$

$1 \longleftrightarrow 0.d_{11}d_{12}d_{13}\ldots$

$2 \longleftrightarrow 0.d_{21}d_{22}d_{23}\ldots$

$3 \longleftrightarrow 0.d_{31}d_{32}d_{33}\ldots$

↖ continue to infinity

$d_{ij} = j^{th}$ decimal place of i^{th} real #

$\frac{1}{2} = 0.500000\ldots$ ↙ never ends

Now consider the number

$d = 0.d_1d_2d_3d_4\ldots$

$$d_k = \begin{cases} 4 & \text{if } d_{kk} \neq 4 \\ 5 & \text{if } d_{kk} = 4 \end{cases}$$

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$1 \leftrightarrow 0.15434\ldots$

$2 \leftrightarrow 0.41198\ldots \quad \Rightarrow \quad 0.445\ldots$

$3 \leftrightarrow 0.97422\ldots$

d is a number in $\mathbb{R}_1$, but d is not in the list! But we claimed all numbers in $\mathbb{R}_1$ are on the list $\rightarrow$ contradiction

If $|A| = |\mathbb{N}|$, we call A countably infinite

If $|A| > |\mathbb{N}|$, we call A uncountably infinite

Q: Let F: the set of functions with domain $\mathbb{N}$ and codomain $\{0,1,2,3,\ldots,9\}$.

Show $|F| > |\mathbb{N}|$

Pf: Suppose for contradiction there is a bijection between F and $\mathbb{N}$

$1 \leftrightarrow f_1 \Rightarrow f_1(1), f_1(2), f_1(3) \ldots$

$2 \leftrightarrow f_2 \Rightarrow f_2(1), f_2(2), f_2(3) \ldots$

$\vdots$

Can create a new function f^* that differs from f_k on input k. That is $f^*(k) \neq f_k(k)$. Thus f^* is not on the list, which contradicts the fact that we have a bijection.