

# Counting Infinite Sets

Given infinite time, a computer can do infinitely many things... but can it do everything? ... will answer in 301. Today, tools.

$$|\mathbb{N}| = \infty$$

$$|\{x > 0 : x \text{ is even}\}| = \infty$$

↙ E } Same infinity?

Q:  $|\mathbb{N}| = |E|$ ?

A)  $|\mathbb{N}| < |E|$

B)  $|\mathbb{N}| = |E|$

C)  $|\mathbb{N}| > |E|$

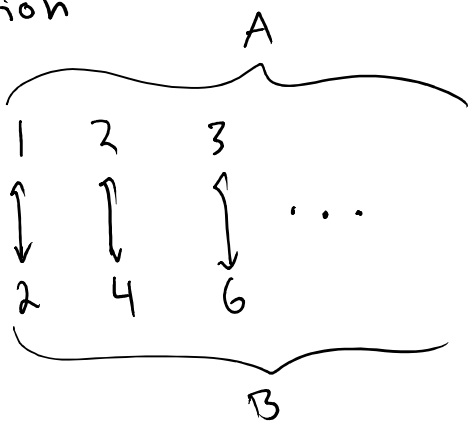


For sets A, B,  $|A| = |B| \iff \exists f: A \rightarrow B$  where f is bijective (surjective & injective)

To prove  $|\mathbb{N}| = |E|$ , use proof by example

Create bijective function:  $f(x) = 2x$ ,  $f: \mathbb{N} \rightarrow E$

Another way to see bijection:



Q: Show  $|\mathbb{N}| = |\{x \in \mathbb{Z} : x > 10\}|$

Show  $|\mathbb{N}| = |\mathbb{Z}|$

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$$f(x) = x + 10$$

$$f: \mathbb{N} \rightarrow \{x \in \mathbb{Z} : x > 10\}$$

1	2	3	4	5	...
↓	↓	↓	↓	↓	
11	12	13	14	15	...

Show  $|\mathbb{N}| = |\mathbb{Z}|$

$$f(x) = \begin{cases} \frac{(x-1)}{2} & x \text{ odd} \\ -\frac{x}{2} & x \text{ even} \end{cases}$$

$$f: \mathbb{N} \rightarrow \mathbb{Z}$$

1	2	3	4	5	...
↓	↓	↓	↓	↓	
0	-1	1	-2	2	...

We call  
any set that has  
the same size as  
 $\mathbb{N}$ , countably infinite

Fact:  $|\mathbb{N}| < |\{x \in \mathbb{R} : 0 < x < 1\}|$   $\mathbb{R}_1$

We call sets that are larger than  $\mathbb{N}$  uncountably infinite

Proof uses Diagonalization:

1. QUIET
2. STONE
3. OFFER
4. CLEAR
5. PHONE

New Word not on list:

<u>S</u>	<u>L</u>	<u>E</u>	<u>E</u>	<u>T</u>
↑	↑	↑	↑	↑
Not	Not	Not	Not	Not
Q	T	F	A	E

Pf: Suppose for contradiction there is a bijection from  $\mathbb{N}$  to  $\mathbb{R}_1$

- 1  $\leftrightarrow$  0. $d_{11}$  $d_{12}$  $d_{13}$ ...
- 2  $\leftrightarrow$  0. $d_{21}$  $d_{22}$  $d_{23}$ ...
- 3  $\leftrightarrow$  0. $d_{31}$  $d_{32}$  $d_{33}$ ...

$d_{ij}$  =  $j^{\text{th}}$  decimal place of  $i^{\text{th}}$  real #

$\frac{1}{2} = 0.500000\dots$   
 ↗ continue to infinity  
 ↘ never ends

Now consider the number

$d = 0.d_1d_2d_3d_4\dots$

$$d_k = \begin{cases} 4 & \text{if } d_{kk} \neq 4 \\ 5 & \text{if } d_{kk} = 4 \end{cases}$$

$$1 \leftrightarrow 0.15434\dots$$

$$2 \leftrightarrow 0.41198\dots$$

$$3 \leftrightarrow 0.97422\dots$$

$$\Rightarrow 0.445\dots$$

$d$  is a number in  $\mathbb{R}_1$ , but  $d$  is not in the list! But we claimed all numbers in  $\mathbb{R}_1$  are on the list  $\rightarrow$  contradiction

If  $|A| = |\mathbb{N}|$ , we call  $A$  countably infinite

If  $|A| > |\mathbb{N}|$ , we call  $A$  uncountably infinite

Q: Let  $F$ : the set of functions with domain  $\mathbb{N}$  and codomain  $\{0, 1, 2, 3, \dots, 9\}$ .

Show  $|F| > |\mathbb{N}|$

Pf: Suppose for contradiction there is a bijection between  $F$  and  $\mathbb{N}$

$$1 \leftrightarrow f_1 \Rightarrow f_1(1), f_1(2), f_1(3) \dots$$

$$2 \leftrightarrow f_2 \Rightarrow f_2(1), f_2(2), f_2(3) \dots$$

$$\vdots$$

Can create a new function  $f^*$  that differs from  $f_k$  on input  $k$ . That is  $f^*(k) \neq f_k(k)$ . Thus  $f^*$  is not on the list, which contradicts the fact that we have a bijection.