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Goals

- Describe sets using set-builder notation
- Translate quantified predicates

Set Builder Notation

$B = \{ f(x) : P(x) \}$ = the set of x where P(x) is true, with f(x) applied to each element

f(x): function of x

P(x): predicate of x

ex: $A = \{ x^2 : x \text{ is even} \} = \{0^2, 2^2, 4^2, 6^2, \dots (-2)^2, (-4)^2, \dots \}$

Repeats not included

$= \{0, 4, 16, 36, \dots \}$

Other ways to represent same set:

$A = \{ x : \frac{\sqrt{x}}{2} \in \mathbb{Z} \}$

Q: Others?

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ex: $A = \{x^2 : x \text{ is even}\}$

Other ways to represent same set:

$$A = \left\{x : \frac{\sqrt{x}}{2} \in \mathbb{Z}\right\}$$

Q: Others?

$$A = \left\{x^2 : \frac{x}{2} \in \mathbb{Z}\right\}$$

$$A = \{(2x)^2 : x \in \mathbb{Z}\}$$

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Quantifiers : turn predicates into statements

Universal Quantifier : $\forall$ means "for all", "every"

ex: $\underbrace{\underbrace{\forall x \in \mathbb{N},}_{\text{quantifier}} \; \underbrace{x > 0}_{\text{Predicate}}}_{\substack{\text{statement} \\ \text{(true)}}}$ $\approx$ "for every natural number x, $x > 0$."

use comma to separate

Existential Quantifier : $\exists$ means "there exists" "there is"

ex: $\underbrace{\underbrace{\exists x \in \mathbb{N}}_{\text{quantifier}} : \underbrace{x < 0}_{\text{predicate}}}_{\substack{\text{statement} \\ \text{(false)}}}$ $\approx$ "there exists a natural number x such that $x < 0$"

use colon "such that" to separate

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Q: Let $M(x,y)$ be the predicate "x is y's parent." Let S be the set of all people who have ever lived

True or false:

$$\forall x \in S, \exists y \in S: M(x,y)$$

True or false?

- $\forall x \in S, \exists y \in S: M(y,x)$
- $\exists x \in S, \forall y \in S: M(x,y)$
- $\exists x \in S, \forall y \in S: M(y,x)$

Q: Let $M(x,y)$ be the predicate that x is y's parent. Let S be the set of all people. True or false:

$$\forall x \in S, \exists y \in S: M(x,y)$$

$\boxed{\text{False:}}$ "Every person has a child"

To determine:

- Imagine you are playing a game against an opponent.
- Opponent: gets to choose $\forall$, wins if makes false
- You: get to choose $\exists$, win if make true
- Both: can choose based on previous person's choice
- Who wins? → True (You)
  → False (Opponent)

ex: $\forall x \in S, \exists y \in S: M(x,y)$

opponent can choose any person. Best strategy: choose x to be someone without a kid

If opponent chooses someone with no kid, we can't find a y to be x's kid. We lose! Statement is false.

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Q: Let $M(x,y)$ be the predicate that x is y's parent. Let S be the set of all people. True or false:

$$\forall x \in S, \exists y \in S: M(x,y)$$

False: "Every person has a child"

Other orders

- $\forall x \in S, \exists y \in S: M(y,x)$ ≡ "Every person has a parent." True
- $\exists x \in S, \forall y \in S: M(x,y)$ ≡ "One person is the parent of all people" False
- $\exists x \in S, \forall y \in S: M(y,x)$ ≡ "One person is the child of all people" False

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* Quantifier must have domain

$\cancel{\forall x, x > 0}$ $\qquad$ $\forall x \in \underbrace{\mathbb{Z}}_{\text{domain = a set}}, x > 0$

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Tips for successful translation

- "All $\underbrace{\text{natural numbers less than 5}}_{\text{subset of } \mathbb{N}}$ divide 12"

$$\forall x \in \mathbb{N},\ \underbrace{x < 5}_{\substack{\text{condition} \\ \text{on subset}}} \rightarrow \underbrace{x \mid 12}_{\text{"x divides 12"}}$$

Common: $\forall x \in \_\_,\ \_\_ \rightarrow \_\_$

- Can combine 2 of same type: $\forall x, y \in \mathbb{N}, \ldots$

- $\underbrace{\exists y \in S}_{\text{"y is quantified"}} : M(x, y) \equiv R(x)$ ← this is a predicate with variable x

"x is not quantified"

$R(\text{Prof K}) = \text{True}$ (I have kid.)

$R(\text{Kora}) = \text{False}$ (Kora doesn't have a kid.)

- Any non-quantified variables become inputs to predicate.
- Statements: all variables must be quantified.