S.KIMMEL

def: Given a sample space S, a random variable X is a function $X: S \to \mathbb{R}$.

ex: Let S be the sample space consisting of all possible outcomes of 4 coin tosses. Let X be the number of heads that occur.

Q: What is $X(T,H,H,H)$? What is $X(T,T,H,T)$?

A: 1, 3 B: 2, 2 $\boxed{\text{C: } 3, 1}$ D: 4, 4

def: The expected or average value of a random variable X is

$$\mathbb{E}[X] = \sum_{i \in S} \Pr(i)\, X(i).$$

Q: From previous example, what is $\mathbb{E}[X]$? (The averge number of heads in 4 coin flips.)

A) 1 $\boxed{\text{B) } 2}$ C) 2.5 D) 4

S. KIMMEL

- I'm guessing you didn't do the following:

$$\mathbb{E}[X] = \sum_{i \in S} \Pr(i) X(i)$$ (2^4 elements of sample space!)

$$\mathbb{E}[X] = \sum_{\substack{i \in S: \\ X(i)=0}} \Pr(i) \cdot 0 + \sum_{\substack{i \in S \\ X(i)=1}} \Pr(i) + \sum_{\substack{i \in S \\ X(i)=2}} \Pr(i) \cdot 2$$

$$+ \sum_{\substack{i \in S \\ X(i)=3}} \Pr(i) 3 + \sum_{\substack{i \in S \\ X(i)=4}} \Pr(i) 4$$

... good practice to finish on your own!

$\Pr(i) = \frac{1}{2^4} = \frac{1}{16}$ in all cases.

$|\{i \in S : X(i) = 0\}| = 1$ $\qquad$ $|\{i \in S : X(i) = 2\}| = \binom{4}{2} = 6$

$|\{i \in S : X(i) = 1\}| = \binom{4}{1} = 4$ $\qquad$ $|\{i \in S : X(i) = 3\}| = \binom{4}{3} = 4$

$|\{i \in S : X(i) = 4\}| = 1$

$$\mathbb{E}[X] = \frac{1}{16}\left(1 \cdot 0 + 4 \cdot 1 + 6 \cdot 2 + 4 \cdot 3 + 1 \cdot 4\right)$$
$$= \frac{1}{16}(32) = 2$$

- Instead you used indicator random variables + linearity of expectation (without knowing!)

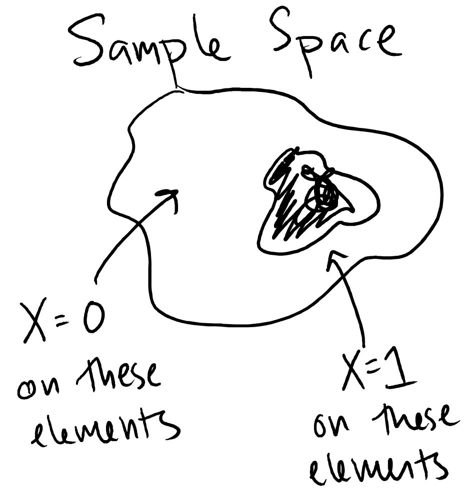
S. KIMMEL

Indicator Random Variable:

def: An <u>indicator random variable</u> X is a random variable such that $X: S \to \{0,1\}$.

An indicator random variable is associated with an event $E \subseteq S$

$$E = \{ i \in S : X(i) = 1 \}$$



$E \subseteq S$ is the event where $X = 1$

Normally write as X_E where

$$X_E(s) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathbb{E}[X_E] = \sum_{i \in S} \Pr(i)\, X_E(i)$$

$$= \sum_{\substack{i \in S: \\ X_E(i) = 0}} \Pr(i) \cdot 0 + \sum_{\substack{i \in S \\ X_E(i) = 1}} \Pr(i)$$

$$= \sum_{i \in E} \Pr(i) = \Pr(E)$$

$$\boxed{\mathbb{E}[X_E] = \Pr(E)}$$

S.KIMMEL

Linearity of Expectation

Let $Y_1, Y_2, \ldots, Y_n$ be random variables on a sample space S. Let $a_1, a_2, \ldots a_n \in \mathbb{R}$. Let Y be a random variable s.t.

$$\forall i \in S, \quad Y(i) = \sum_{k=1}^{n} a_k Y_k(i)$$

distribute $\mathbb{E}$ inside summation

Then

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{k=1}^{n} a_k Y_k\right] = \sum_{k=1}^{n} a_k \, \mathbb{E}[Y_k]$$

Ex: Let X_k be the indicator random variable that takes value 1 if k^{th} coin flip is Heads.

$$X = \#\text{ of heads in } 4 \text{ coin tosses}$$

Then $X = \sum_{k=1}^{4} X_k$

e.g.
$$X(H,T,T,H) = \overbrace{X_1(H,T,T,H)}^{1} + \overbrace{X_2(H,T,T,H)}^{0} + \underbrace{X_3(H,T,T,H)}_{0} + \underbrace{X_4(H,T,T,H)}_{1} = 2$$

$$\mathbb{E}[X] = \sum_{k=1}^{4} \mathbb{E}[X_k] = \sum_{k=1}^{4} \Pr(k^{th} \text{ flip is Heads})$$

$$= \sum_{k=1}^{4} \frac{1}{2} = 2$$

linearity of expectation

indicator random variable property