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Goals

- Describe proof by contrapositive, iff proofs, Proof by cases, proof by example.
- Practice writing proofs

✱ Doing self-assessment + unsure if correct... contact assigned TA!

Proof By Contrapositive

- Use: Prove $P \to Q$

$$P \to Q \equiv \neg Q \to \neg P$$

P	Q	$\neg Q$	$\neg P$	$P \to Q$	$\neg Q \to \neg P$
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

Structure:

We prove the contrapositive. Assume $\neg Q$. Explain explain, explain. Therefore, $\neg P$.

- Use: Prove $\forall x \in S,\ P(x) \to Q(x)$

Structure

We prove the contrapositive
Let $x \in S$ and assume $\neg Q(x)$

Explain, explain, explain.

Therefore, $\neg P(x)$.

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Q: What is the first sentence of a contrapositive proof of:

"If a^2 is not divisible by 4, then a is odd."

A) We prove the contrapositive. Assume a is odd.

B) We prove the contrapositive. Assume a is even.

C) We prove the contrapositive. Assume $\neg 4 \mid a^2$.

D) We prove the contrapositive. Assume $4 \mid a^2$.

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Q: What is the first sentence of a contrapositive proof of:

$$\text{"If } \overbrace{[a^2 \text{ is not divisible by } 4]}^{P} \text{ then } \overbrace{[a \text{ is odd}]}^{Q}\text{"}$$

$P \rightarrow Q$

A) Assume a is odd

$\boxed{\text{B) Assume } a \text{ is even}}$

C) Assume $\neg 4 \mid a^2$

D) Assume $4 \mid a^2$

$$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$$

$$\neg P = \text{"}a \text{ is even"}$$

For practice, complete the proof. (Solution on next page.)

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optional

$\Downarrow$

We prove the contrapositive. [Let $a \in \mathbb{Z}$.] Suppose a is even. Then $\exists k \in \mathbb{Z} : 2k = a$. This means $a^2 = 4k^2$. Since k^2 is an integer, $4 | a^2$.

* There is an implied "for all" in the proof statement. (otherwise it is a predicate & we can't prove true or false). Therefore, we need "Let $a \in \mathbb{Z}$."

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Iff Proofs

Iff = "if and only if"

Use: $P \leftrightarrow Q$

$(P \to Q) \wedge (Q \to P) \equiv P \leftrightarrow Q$ (use truth table proof)

Structure

P For the forward direction, [Proof of $P \to Q$]

P For the backwards direction, [Proof of $Q \to P$]

Could be direct or contrapositive

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Proof By Cases

Use: $\forall x \in A, \ldots$

Want to prove for all x in A, but sometimes need different techiques for different subsets of A

ex: $\forall x \in \mathbb{Z} \ldots$ where to start.?

$x > 0$ $x < 0$

Split into cases.

$\Downarrow$

Now can turn each case into math and move forward.

$\Uparrow$

Missing case $x = 0$

Make sure cases contain all possibilities

Better

$\forall x \in \mathbb{Z}$

$x > 0$ $x = 0$ $x > 0$

"Let $x \in \mathbb{Z}$. There are three cases: $x > 0$, $x < 0$, and $x = 0$.

For the first case, assume $x > 0 \ldots$

¶ For the second case, assume $x < 0 \ldots$

¶ For the third case, assume $x = 0 \ldots$

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When proving correctness of algorithm and see

if ..

else ..

$\Big\}$ use proof by cases to show

- cover all situations
- behaves correctly in all situations

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Proof by Contradiction

Use: any statement P

Proof has two parts

① (Direct) $\neg P \to Q$

② (Direct) $\neg P \to \neg Q$

$\therefore P$

Most common

Structure

For contradiction assume $\neg P$

Explain explain... Q

Explain explain... $\neg Q$, a contradiction.

Thus, P is true

When start, don't know what Q is... you need to keep your eye out for what might be the contradiction.

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Q: Prove $\sqrt{2}$ is irrational

* Not of form $P \to Q$

A: For contradiction, suppose $\sqrt{2}$ is rational. Then $\exists\, a, b \in \mathbb{Z} : \frac{a}{b} = \sqrt{2}$ where the fraction is fully simplified, so $\neg \exists\, c \in \mathbb{Z} : c|a \wedge c|b$.
Squaring both sides, we have

$$a^2 = 2b^2.$$

Thus $2|a^2$. But we've previous proved this implies $2|a$. This means $\exists\, m \in \mathbb{Z} : 2m = a$. Plugging in, we have

$$4a^2 = 2b^2.$$

Dividing by 2, we get

$$2a^2 = b^2.$$

But this means $2|b^2$, and so $2|b$. But this means $2|a$ and $2|b$, which contradicts the fact that $\frac{a}{b}$ is fully simplified. $\square$

Q: Prove: $\neg \exists\, x, y \in \mathbb{Z} : x^2 = 4 \quad + 2$