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Relations

Relation is a generalization of a graph

$$E \subseteq V \times V$$

edges are a subset of all possible pairs of vertices

Relation $R \subseteq A \times B$ (for sets A, B)

"Relation R on A" means $R \subseteq A \times A$

Q: Which of the following relations on $\mathbb{Z}$ has $(1,-1)$ as an element?

A) $\{(a,b) : a^2 = b\}$

B) $\{(a,b) : a = b^2\}$

C) $\{(a,b) : ab = 1\}$

D) None of above

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Q: Which of the following relations on $\mathbb{Z}\times\mathbb{Z}$ has $(1,-1)$ as an element?

A) $\{(a,b) : a^2 = b\}$ ← contains $(-1,1)$

$\boxed{\text{B)}}$ $\{(a,b) : a = b^2\}$ ← $1 = (-1)^2$
Order matters

C) $\{(a,b) : ab = 1\}$

D) None of above

$$\{(a,b) : a = b^2\} = \{(0,0), (1,-1), (1,1), (4,2), (4,-2) \ldots\}$$

3 Properties of Relations on A

- Reflexive: $\forall a \in A, (a,a) \in R$
  - So $\{(a,b) : a = b^2\}$ is not reflexive because $(2,2) \notin R$ (Proof by counter example "not all")
- Symmetric: $\forall a,b \in A, (a,b) \in R \rightarrow (b,a) \in R$
  - So $\{(a,b) : a = b^2\}$ is not symmetric because $(1,-1) \in R$ but $(-1,1) \notin R$

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- **Transitive**: $\forall a,b,c \in A, \big((a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R \big)$

  So $\{(a,b) : a = b^2\}$ is not transitive because

  $(16,4) \in R \wedge (4,2) \in R$ but $(16,2) \notin R$

Equality "=" is Reflexive, Symmetric, Transitive:

- $a = a$ (Reflexive)
- $a = b \rightarrow b = a$ (Symmetric)
- $(a = b \wedge b = c) \rightarrow a = c$ (Transitive)

Equivalence Relation is Reflexive, Symmetric, Transitive

If R is an equivalence relation. We can interpret $(a,b) \in R$ as "a is equivalent to b."

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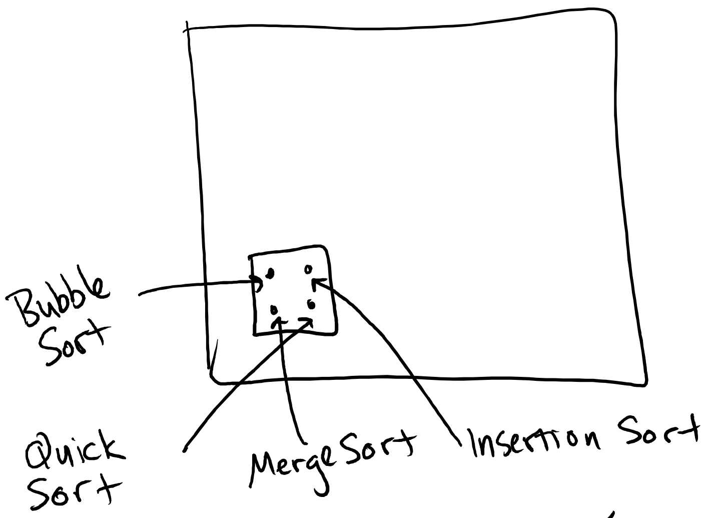
Equivalence Relations partition sets

← divide into non-intersecting subsets



ex: A = set of all algorithms

$R \subseteq A \times A$

$R = \{(a,b)$: a and b give the same output given the same input $\}$

All these sorting algorithms are equivalent under R

They form an <u>equivalence class</u>

$$\Rightarrow a_R = \{ b : (a,b) \in R \}$$

set of all elements of A that are equivalent to a under R.

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Q: What is an equivalence class for the equivalence relation $L = \{(a,b) : \text{length}(a) = \text{length}(b)\} \subseteq S \times S$

set of all bit strings

A) $(00, 11)$

B) $\{(0,0), (0,1), (1,0), (1,1)\}$

C) $\{00, 01, 10, 11\}$

D) It's not an equivalence relation, so can't get equivalence classes

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Q: What is an equivalence class for the equivalence relation $L = \{(a,b) : \text{length}(a) = \text{length}(b)\} \subseteq S \times S$

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D) It's not an equivalence relation, so can't get equivalence classes

Equivalence classes are sets of bitstrings with the same length

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A) Equivalence Relation

B) Not reflexive

C) Not symmetric

D) Not transitive

1. $$R = \{(a,b) \in \mathbb{R} \times \mathbb{R} : a - b \in \mathbb{Z}\}$$

A) <u>Equivalence Relation</u>

- Reflexive: Let $a \in \mathbb{R}$. Then $a - a = 0$, and $0 \in \mathbb{Z}$, so $(a,a) \in R$.
- Symmetric: Let $a, b \in \mathbb{R}$. Assume $(a,b) \in R$. Then $a - b \in \mathbb{Z}$, so $b - a \in \mathbb{Z}$, so $(b,a) \in R$.
- Transitive: Let $a, b, c \in \mathbb{R}$. Assume $(a,b), (b,c) \in R$. Then $\exists\, x, y \in \mathbb{Z} : a - b = x \wedge b - c = y$. Thus $a - c = x + y$ which is an integer, so $(a,c) \in R$

$\Rightarrow$ Equivalence Classes: $X_R = \{y + X : y \in \mathbb{Z}\}$

ex: $.7_R = \{\ldots -1.3, -.3, .7, 1.7, \ldots\}$