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Recurrence & Time Complexity

When an algorithm has loops, we have tools for analyzing time complexity. What about recursive algorithms?

```
Factorial(n)
1. if(n==1): return 1
2. else: return n · Factorial(n-1)
```

$T(n)$ is time complexity of Factorial(n)

- $T(n) = C$ if $n = 1$ ← Initial conditions
- $T(n) = D + T(n-1)$ if $n \geq 1$ ← Recurrence Relation

} Recurrence Relation

Some constant

D: constant # of operations in this call

$T(n-1)$: + all operations done in recursive call

e.g. $T(3) = D + T(2)$

$T(3) = D + D + T(1) = 2D + C$

* Sometimes can have multiple initial conditions

$T(n) = T(n-1) + T(n-2)$ ← if recursive relation depends on more than just $n-1$

$T(1) = A \quad T(2) = B$

$T(3) = A + B$

Solving a Recurrence Relation (iterative method)

$$T(n) = D + T(n-1)$$
$$= D + D + T(n-2)$$
$$= D + D + D + T(n-3)$$

Looks like pattern is

$$T(n) = kD + T(n-k)$$

We know $T(1) = 1$

$n-k=1 \Rightarrow n-1=k$ Plug in k:

$$T(n) = (n-1)D + T(1) = nD + C - D = O(n)$$

Not really rigorous... Instead:

1. Use this process to guess solution
2. Use (strong) induction to prove.

We'll hopefully get back to this next later

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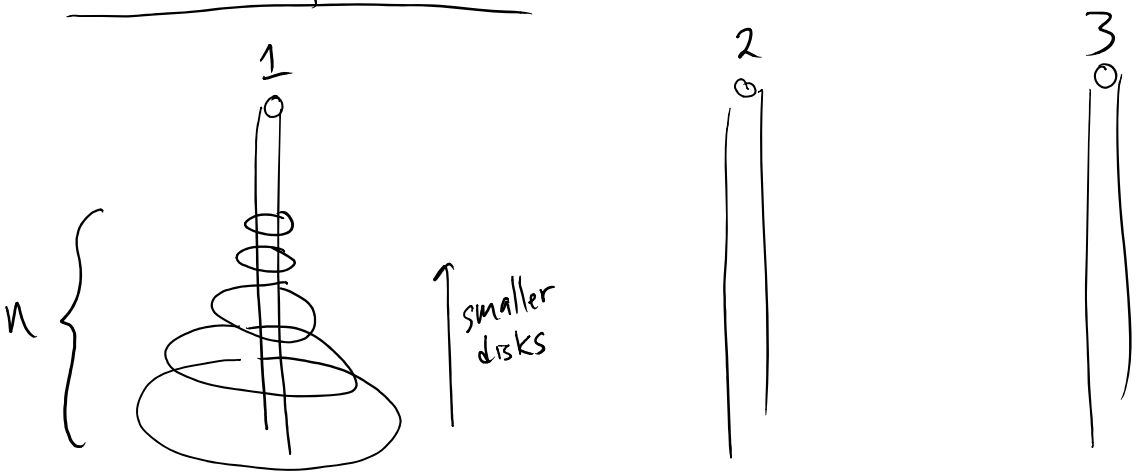
Another example...

Towers of Hanoi



Task: Move Tower from 1 to 3 without putting larger disk on top of smaller.

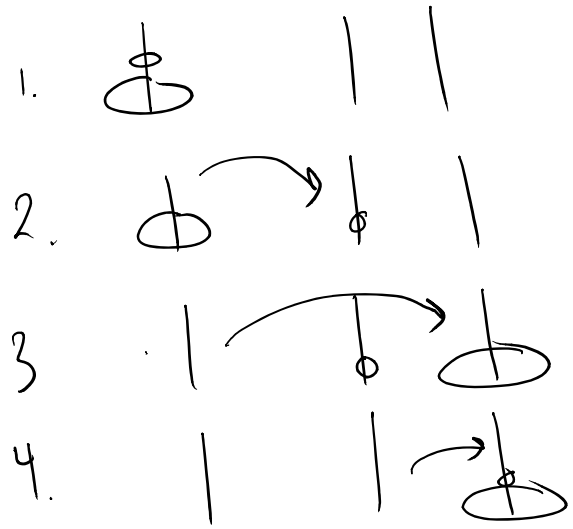
Q: How many moves do you need?

ex 1.

2.

3.

4.



n=2 → 3 moves

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Q: Which of the following recurrence relations is correct if $T(n)$ = # of moves w/ n disks

A) $T(1) = 1$; $T(n) = 2 + T(n-1)$

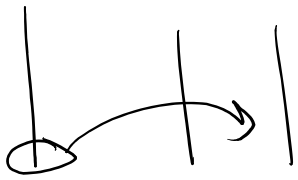
B) $T(2) = 1$; $T(n) = 2T(n-1)$

C) $T(1) = 1$; $T(n) = 2T(n-1)$

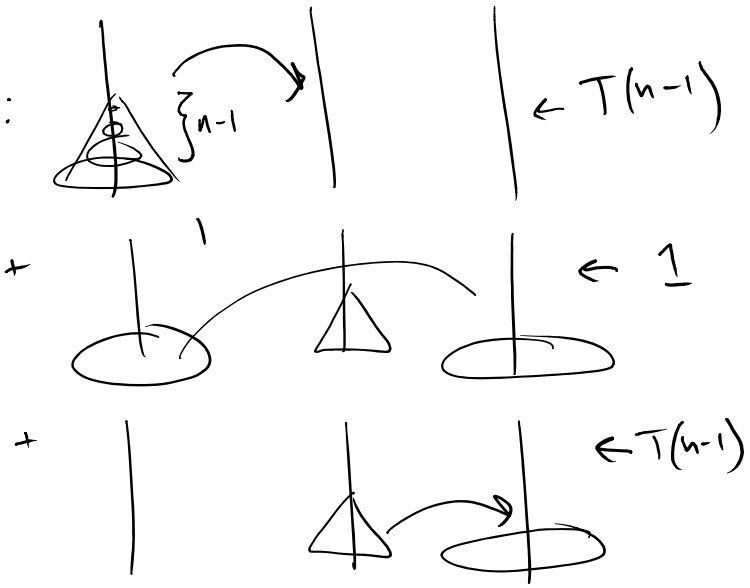
D) $T(1) = 1$; $T(n) = 2T(n-1) + 1$

Answer D:

$T(1) = 1$



$T(n)$:



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Solve for $T(n)$ using iterative method

$$T(n) = 2T(n-1) + 1$$

$$= 2\left(2 \cdot T(n-2) + 1\right) + 1 = 4 \cdot T(n-2) + 2 + 1$$

$$= 4 \cdot \left(2 \cdot T(n-3) + 1\right) + 2 = 8 \cdot T(n-3) + 4 + 2 + 1$$

$$= 8\left(2 \cdot T(n-4) + 1\right) + 4 + 2 = 16 \cdot T(n-4) + 8 + 4 + 2 + 1$$

$$\vdots$$

$$= 2^k T(n-k) + \sum_{j=1}^{k-1} 2^j$$

Base case: $n - k = 1 \Rightarrow k = n-1$

$$= 2^{n-1} T(1) + \sum_{j=0}^{n-2} 2^j$$

$$= \sum_{j=0}^{n-1} 2^j = 1 + 2 + 4 + 8 + 16 + 32 + \cdots + 2^{n-2} = \frac{2^n - 1}{2 - 1}$$

Will prove in HW

$$= 2^n - 1 = O(2^n)$$

Worksheet!