

Probability

Probability is all about counting

ex: What is the probability that the outcome of a die roll is at least 5?

1. Count all outcomes

$$|\{1, 2, 3, 4, 5, 6\}| = 6$$

2. Count outcomes where rolled value is ≥ 5

$$|\{5, 6\}| = 2$$

3. Take ratio:

$$\frac{2}{6} = \frac{1}{3}$$

Terminology

- Sample space: set of all possible outcomes
- Event: subset of sample space.

def: If all elements in sample space S are equally likely, the probability of an event E is

$$\Pr(E) = \frac{|E|}{|S|}$$

Q: Lottery where a 4-digit number is chosen randomly. What is the size of the sample space?

A) 40 B) 10,000 C) $\binom{10}{4}$

↑

Use product rule!

$$10 \cdot 10 \cdot 10 \cdot 10$$

Q. Suppose you win some money if you get 3 of 4 numbers matching.

ex: Lot~~to~~: 5 7 0 1

You: 5 7 5 1

What is the probability you get 3 out of 4 correct?

A) $3/10^4$ B) $10/10^4$ C) $36/10^4$ D) $40/10^4$

↑

Sum rule: $9 + 9 + 9 + 9 = 36$

or Product rule: $\binom{4}{3} \cdot 9 = 36$

Note : Probability that an event E does NOT happen

$$1 - P(E)$$

(Try to prove this using definition of probability!)

Q. What is the probability that you DON'T win lottery?
(Don't get 3 or 4 digits the same)

A) 0.9962 B) 0.9963 C) 0.9964 D) 0.9965

↑

Sum rule

$$\frac{36 \text{ ways to match 3 #'s} + 1 \text{ way to match 4 #'s}}{37 \text{ ways to win}}$$

$$\Pr(\text{NOT win}) = 1 - \frac{37}{10000} = 0.9963$$

Probability of Union of Events

(subtraction rule)

$$\begin{aligned} \Pr(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\ &= \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2) \end{aligned}$$

What happens if all outcomes are not equally likely?

ex: Weighted coin: 25% Heads, 75% Tails

3 coin flips

1. What is sample space of 2 coin flips?

$$\{HHH, HHT, HTH, HTT, THT, THT, TTH, TTT\}$$

2. What is size of sample space? 8

Unequal Probability doesn't affect sample space

In general

Given a sample space S , can create a probability function (distribution) such that

$$\bullet \forall s \in S, 0 \leq \Pr(s) \leq 1$$

$$\bullet \sum_{s \in S} \Pr(s) = 1$$

Then the probability of event $E \subseteq S$ is

$$\Pr(E) = \sum_{s \in E} \Pr(s)$$

Suppose 6 is twice as likely to be rolled as every other die outcome.

• What is the probability of 6?

• What is the probability of ≥ 5 ?

$$\Pr(1) + \Pr(2) + \Pr(3) + \Pr(4) + \Pr(5) + \Pr(6) = 1$$

$$x + x + x + x + x + 2x = 1$$

$$x = \frac{1}{7}$$

$$\Pr(6) = 2x = \frac{2}{7}$$

$$\Pr(5 \text{ or } 6) = 3x = \frac{3}{7}$$