Today: Quantum States & Measurements

Last time: Questions about quant. Crypto?

Representing States with Vectors:

\[
\begin{align*}
\uparrow &= |0\rangle \\
\downarrow &= |1\rangle \\
\frac{1}{\sqrt{2}} |1\rangle &= |+\rangle \\
\frac{-1}{\sqrt{2}} |1\rangle &= |\rightarrow\rangle
\end{align*}
\]

Circularly polarized, used for 3D movies:

\[
\begin{align*}
\frac{1}{\sqrt{2}} |i\rangle &= |\rightarrow\rangle \\
\frac{-1}{\sqrt{2}} |i\rangle &= |\leftarrow\rangle
\end{align*}
\]

Qubit States

\[
(a_0, a_1) \in \mathbb{C}, \quad |a_0|^2 + |a_1|^2 = 1
\]

The bit has two states, 0, 1. The qubit has two "standard basis" states:

\[|0\rangle = "\text{zero state}" = |0\rangle \quad "\text{Ket notation}" \]

\[|1\rangle = "\text{one state}" = |1\rangle\]
Common to write
\[ |+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \]

"Superposition" = linear combination

"|+\rangle is in a superposition of \(|0\rangle\) and \(|1\rangle\)"

Note: \(|+\rangle, |-\rangle\)

\[ \text{basis} \]

So
\[ |0\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle) \]

"|0\rangle is in a superposition of \(|+\rangle\) and \(|-\rangle\)"

Whether a state is a superposition or not depends on your basis. However, when we say superposition, we usually mean relative to standard basis.
**Bra & Kets**

In Linear Algebra class: \[ X = \begin{pmatrix} x_0 & \cdots & x_k \\ \vdots & \ddots & \vdots \\ x_k & \cdots & x_0 \end{pmatrix} \quad \xrightarrow{\text{complex conjugate}} \quad X^* = (x_0^*, x_1^*, \ldots, x_k^*) \]

In Quantum computing: \[ |\psi\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \quad \langle \psi | = (a_0^*, a_1^*) \]

"ket \psi_i" \leftrightarrow "bra \ psi_i"

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**Inner Product**

\[ |\psi\rangle = a_0 |0\rangle + a_1 |1\rangle \]
\[ |\phi\rangle = b_0 |0\rangle + b_1 |1\rangle \]

\[ \langle \phi | \psi \rangle = \begin{pmatrix} a_0^* & b_0^* \\ a_1^* & b_1^* \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = b_0^* a_0 + b_1^* a_1 \]

"inner product"

Using bra/kets: \[ \langle \phi | \psi \rangle = (b_0^* \langle 0 | + b_1^* \langle 1 |)(a_0 |0\rangle + a_1 |1\rangle) = a_0^* b_0 \langle 0 | 0 \rangle + a_0^* b_1 \langle 0 | 1 \rangle + b_1^* a_0 \langle 1 | 0 \rangle + b_1^* a_1 \langle 1 | 1 \rangle \]

Whenever you see inner product of standard basis states, is 0 if different, 1 if same.

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**Q:** For a qubit state \[ |\psi\rangle = a_0 |0\rangle + a_1 |1\rangle \], what is \[ \langle \psi | \psi \rangle ? \]

\[ \langle \psi | \psi \rangle = \begin{pmatrix} a_0^* & a_1^* \\ a_0 & a_1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = a_0^* a_0 + a_1^* a_1 = |a_0|^2 + |a_1|^2 = 1 \]

Inner product of quantum state with itself is 1
2. Qubit Measurement (von Neumann Measurement)

Measurement is represented by an orthonormal basis: \( \mathcal{M} = \{ |\phi_0\rangle, |\phi_1\rangle \} \)

- 2 states \( |\phi_0\rangle, |\phi_1\rangle \) st. \( \langle \phi_i | \phi_j \rangle = 0 \) and \( \langle \phi_0 | \phi_0 \rangle = \langle \phi_1 | \phi_1 \rangle = 1 \)

If initially, system is \( |\psi\rangle \),
- with probability \( |\langle \phi_0 | \psi \rangle|^2 \), get outcome \( |\phi_0\rangle \), state becomes \( |\phi_0\rangle \)
- with probability \( |\langle \phi_1 | \psi \rangle|^2 \), get outcome \( |\phi_1\rangle \), state becomes \( |\phi_1\rangle \)

We say "state \( |\psi\rangle \) collapses to \( |\phi_0\rangle \) or \( |\phi_1\rangle \)."