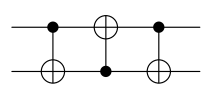
CS333 - Problem Set 6

Due: Wednesday, April 11 th before class

See final page for hints.

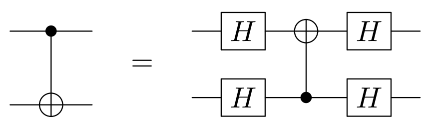
1. CNOT properties:

   (a) **[6 points]** What does the following circuit do? (Your answer should be a simple description in English, not math.)

   (1)

   (b) **[6 points]** Prove the following two circuits are equal. Use the ket-bra description of CNOT to do this analysis, and note that $X = |+\rangle\langle+| - |-\rangle\langle-|$:

   (2)

   (c) **[6 points]** (** Big picture **) Describe some big picture properties of the quantum CNOT gate (based on this problem or past problems or from class)

2. (** Big picture **) In class, I said that you can do universal quantum computation with a circuit which always starts with n qubits prepared as $|0\rangle^{\otimes n}$, applies a unitary U, and always measures each qubit in the standard basis. However, perhaps the optimal computation calls for starting with a state $|\psi\rangle \neq |0\rangle^{\otimes n}$, applying a unitary V, and then measuring in a basis $\{|\phi_0\rangle, |\phi_1\rangle, \dots, |\phi_{2^n-1}\rangle\}$. In this question, you will describe how you can implement the ideal circuit by starting with the state $|0\rangle^{\otimes n}$, applying a unitary V_{prep}, then the unitary V, and then a unitary V_{meas}, followed by a measurement in the standard basis. In other words, the unitary we apply between state preparation and final measurement is $V_{meas} V V_{prep}$. (For this question, ignore the fact that we can only approximately implement any unitary - instead assume that we can exactly implement any unitary.)

   (a) **[6 points]** Please give a description of a unitary V_{prep}, and explain why it works.

   (b) **[6 points]** Please give a description of a unitary V_{meas}, and explain why it works.

3. Let $f : \{0,1\}^2 \to \{0,1\}$ be a black-box function whose input is two bits, where f takes the value 1 on exactly one input value (and is zero on the other three input values). The goal of the one-out-of-four search problem is to find the unique $(x_1, x_2) \in \{0,1\}^2$ such that $f(x_1, x_2) = 1$.

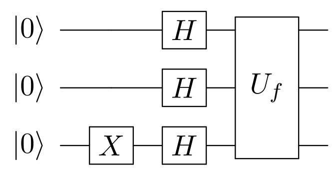
   (a) **[3 points]** Write the truth tables of the four possible functions f. (Label the functions f_{00}, f_{01}, f_{10}, f_{11} according to the position of the one-valued input.)

(b) **[6 points]** How many classical queries are needed to solve one-out-of-four search in the worst case?

(c) **[6 points]** Suppose f is given as a quantum black box U_f acting as

$$|x_1, x_2\rangle|y\rangle \stackrel{U_f}{\rightarrow} |x_1, x_2\rangle|y \oplus f(x_1, x_2)\rangle. \tag{3}$$

Determine the output of the following quantum circuit for each of the possible functions f:



(4)

(d) **[6 points]** Show that the four possible outputs obtained in the previous part are all orthogonal to each other. What does this tell you about the quantum query complexity of one-out-of-four search?

(e) **[6 points]** (** Big Picture **) Describe at a high level (using words like phase kickback and superposition) how this quantum algorithm works.

Hints!

1. (a) Recall a unitary (in this case the unitary is a circuit) is completely determined by its action on an orthonormal basis. What does this unitary do to the standard basis? Then use linearity to figure out what the circuit does to a superposition. (Linearity means $U(|x\rangle + |y\rangle = U|x\rangle + U|y\rangle))$

   (b) We can represent CNOT in bra ket notation as: $|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$

2. (There is not a unique unitary for either problem.) Describe the unitary as a sum of ket-bra terms. Think about what states you would like to transform, and how you can represent those as part of an orthonormal basis.

3. (a)

   (b) 3 (why?)

   (c) Show that before U_f, the state is:

$$\frac{1}{2\sqrt{2}} \sum_{x_1, x_2 \in \{0,1\}} |x_1, x_2\rangle \left(|0\rangle - |1\rangle\right). \tag{5}$$

   Then determine how U_f acts on each term in the superposition. (Think phase kick-back!)

   (d)