CS333 - Problem Set 4

Due: Wednesday, Mar. 14

1. Consider the entangled 2-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}|01\rangle_{AB} - \frac{1}{\sqrt{2}}|10\rangle_{AB}$. This state has some strange properties - in particular the two qubits are perfectly anticorrelated. It's behavior is so strange, it caused Einstein to believe that quantum mechanics couldn't possibly describe reality.

   (a) Show that if Alice and Bob each measure their qubit using the measurement $M(\eta) = \{|\phi_0(\eta)\rangle, |\phi_1(\eta)\rangle\}$, where $|\phi_0(\eta)\rangle = \cos(\eta)|0\rangle + \sin(\eta)|1\rangle$ and $|\phi_1(\eta)\rangle = -\sin(\eta)|0\rangle + \cos(\eta)|1\rangle$, then if Alice gets outcome $|\phi_0(\eta)\rangle$, Bob will get outcome $|\phi_1(\eta)\rangle$, or vice versa.

   (b) What is the probability that Alice gets either outcome $|\phi_0(\eta)\rangle$ or $|\phi_1(\eta)\rangle$.

   (c) This anticorrelation is strange because of the following thought experiment. Suppose Alice took her qubit to the moon, and Bob stayed on Earth. Now Alice performs her measurement first, and suppose she gets outcome $|\phi_0(\eta)\rangle$. Then, even if Bob performs his measurement before any lightspeed communication can have happened between Alice and Bob, Bob's qubit somehow knows to choose the opposite outcome. People thought that this might enable faster than light communication, but explain why Part (b) of this question rules out faster than light communication.

   Hint No hints :(

2. Eve knows it is impossible to create a generic cloner, but she thinks she has found a pretty good cloner that will help her break the BB84 cryptography scheme. For each round of the cryptography protocol, she prepares a qubit in the state $|0\rangle_E$. When the photon that Alice is sending to Bob comes to her, she applies the unitary:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{1}$$

   to the combined state of $|\psi\rangle|0\rangle_{AE}$ (where $|\psi\rangle_A$ is the state of Alice's photon.) Then after acting with the unitary, she sends qubit A to Bob, and keeps system B but doesn't measure it yet. When Alice and Bob announce their measurement bases, if Alice and Bob's measurement bases were the same, Eve measures her system in that basis. Is this a good strategy?

   Hint This strategy is not better than the strategies where Even just immediately measures. (But why?)

3. Alice, Bob, and Charlie are playing a game where the referee sends them each a qubit that is part of a 3-qubit state. The referee promises that the state is one of two options: $|\psi_0\rangle$ or $|\psi_1\rangle$. Alice, Bob, and Charlie can each make a local measurement on their part of the state (i.e. they can make a quantum measurement on their qubit). After the measurement, they can communicate (say on the phone), discuss their outcomes, and try to decide if the state is $|\psi_0\rangle$ or $|\psi_1\rangle$. For each of the following, either give a strategy such that Alice, Bob, and Charlie can always win, or explain why they will always lose with some probability.

   (a)
$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|000\rangle_{ABC} + |111\rangle_{ABC})$$
$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|010\rangle_{ABC} + |101\rangle_{ABC}) \tag{2}$$

   (b)
$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|000\rangle_{ABC} + |111\rangle_{ABC})$$
$$|\psi_1\rangle = |000\rangle_{ABC} \tag{3}$$

   (c)
$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|000\rangle_{ABC} + |111\rangle_{ABC})$$
$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|000\rangle_{ABC} - |111\rangle_{ABC}) \tag{4}$$

   Hints

   (a) Possible
   (b) Impossible (why?)
   (c) Possible

4. Not all entangled states are equal. Some states are more entangled than others. In this problem, you will investigate how the level of entanglement in Alice and Bob's shared state affects their ability to win the CHSH game. (Warning - this problem is calculation heavy, but good practice!)

   (a) Let $|\psi(\theta)\rangle$ be the two-qubit state $|\psi(\theta)\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$. For what values of θ is $|\psi(\theta)\rangle$ entangled? (Please only consider $0 \leq \theta \leq \pi/2$.)
   (b) Suppose Alice and Bob share a state $|\psi(\theta)\rangle$ as a resource, and use the same strategy we discussed in class to try to win the CHSH game. What is Alice and Bob's average probability of winning the game, as a function of θ, where the average is taking over the possible values of x and y? (Hint - You can halve the number of calculations you have to do by just calculating the probability of the winning outcomes.)
   (c) Do Alice and Bob always do better than the best classical strategy, as long as they have some entanglement? Or do they need a certain amount of entanglement (certain values of θ)?

Hints

(a) No hint.

(b) Their probability of winning is (unless I've made an algebra error!)

$$\frac{4+\sqrt{2}(1+\sin(2\theta))}{8}. \tag{5}$$

(c) No Hint