CS333 - Problem Set 2

Due: Wed, Feb 28

1. **[0 points - (Optional, if want basic practice)]** Decide whether each of the following vectors could represent a qubit state. If not, find a real number such that multiplying the vector by the number will create a valid quantum qubit state.

   (a) $\begin{pmatrix} e^{i\xi}\cos(\theta) \\ e^{-i\phi}\sin(\theta) \end{pmatrix}$

   (b) $\frac{1}{2}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$

2. **[0 points - (Optional, if want basic practice)]** Does the following represent a valid qubit measurement? Why or why not?

$$M = \left\{ \sqrt{1/3}|0\rangle + i\sqrt{2/3}|1\rangle, \sqrt{2/3}|0\rangle + i\sqrt{1/3}|1\rangle \right\} \tag{1}$$

3. Let $M = \{|\phi_0\rangle, |\phi_1\rangle\}$ be an orthonormal basis representing a qubit measurement, and let $|\psi\rangle$ be a vector representing a qubit quantum state.

   (a) **[6 points]** There exist $\alpha_0, \alpha_1 \in \mathbb{C}$ such that $|\psi\rangle = \alpha_0|\phi_0\rangle + \alpha_1|\phi_1\rangle$. Show that $|\alpha_0|^2 + |\alpha_1|^2=1$.

   (b) **[3 points]** Suppose we measure $|\psi\rangle$ using M. Let p_0 be the probability of outcome $|\phi_0\rangle$ and let p_1 be the probability of outcome $|\phi_1\rangle$. Use part (a) to show that $p_0 + p_1 = 1$, that is, the sum of the outcome probabilities is 1.

   (c) **[3 points]** What does this problem tell you about quantum measurements and quantum states? (In other words, why did I have you do this problem?)

4. Let $|\psi\rangle$ be a vector representing a qubit quantum state. Let $|\psi'\rangle = e^{i\phi}|\psi\rangle$ for $\phi \in \mathbb{R}$.

   (a) **[3 points]** Show that $|\psi'\rangle$ also represents a qubit state.

   (b) **[3 points]** Show that any measurements give exactly the same outcome statistics and states on $|\psi\rangle$ and $|\psi'\rangle$.

   (c) **[3 points]** Is it possible to tell the difference between $|\psi\rangle$ and $|\psi'\rangle$? What does this mean? What is the significance of this problem?

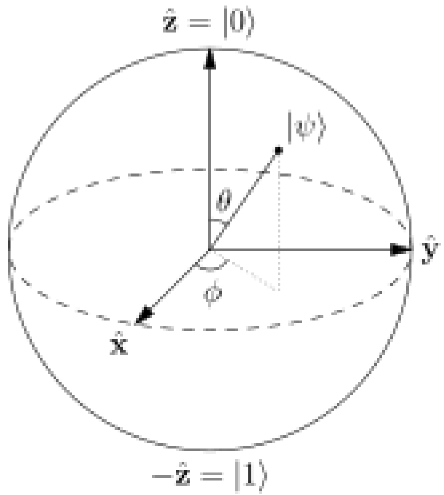
5. **[6 points]** The Bloch sphere is a useful tool for visualizing single qubit states. In the last problem, we saw that a global phase has no effect on the state. Using this degree of freedom,

along with with normalization condition, we can parameterize all single qubit states using two parameters:

$$|\psi(\theta, \phi)\rangle = \begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix} \tag{2}$$

where $\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi)$.

Now the surface of a sphere is parameterized by the polar angle $\theta \in [0, \pi]$ and the azimuthal angle $\phi \in [0, 2\pi)$. Thus there is a one-to-one correspondance between a single qubit states and points on the surface of the sphere: we identify the state $|\psi(\theta/2, \phi)\rangle$ with the point on the sphere with polar angle θ and azimuthal angle ϕ, as in the following diagram. (**Important:** the θ in Eq. (2) is not the same as the θ in the figure of the sphere; they differ by a factor of 2, so be careful of this when moving from one represenation to another. However, θ's are the same in the equation and figure.)



You can verify that the vector $\hat{z}$ (north pole direction) corresponds to $|0\rangle$, and $-\hat{z}$ (south pole direction) correspons to $|1\rangle$. What states do the vectors $\hat{x}$, $\hat{y}$, $-\hat{x}$, and $-\hat{y}$ correspond to? ($\hat{y}$ is 90° from $\hat{x}$ on the equator.) What is the absolute value of the inner product squared of any two states that are at 90° from each other? 180° from each other?

6. **[6 points]** Using kets, bras, or other linear algebraic representations of quantum states and measurements please describe what possible events might occur, and calculate the probability of those events in the following scenario:

   - Alice prepares a horizontally polarized photon and sends to Bob.
   - Eve intercepts the photon and has it pass through a vertically polarized filter before trying to detect the photon. If she detects a photon, she prepares a vertically polarized photon to send to Bob, and otherwise, she sends Bob a horizontally polarized photon.
   - Bob measures the photon he received from Eve by putting a right diagonally polarized filter in front of his photon detector.

7. (* = challenge) Let θ be a fixed, known angle. Suppose someone flips a fair coin and,

depending on the outcome, either gives you the state

$$|0\rangle \qquad \text{or} \qquad \cos\theta|0\rangle + \sin\theta|1\rangle \tag{3}$$

(but does not tell you which). Describe a qubit measurement for guessing which state you were given, succeeding with as high a probability as possible. Also indicate the success probability of your procedure. (You do not need to prove that your procedure is optimal.)

8. How long did you spend on this homework?