1 Complex Numbers

A complex number is written as $a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. It is useful to think about complex numbers as points on a plane:

If $x = a + bi$, then the complex conjugate of $x$, denoted $x^*$, is $a - bi$. If $w$ and $x$ are complex numbers, then

\[
(wx)^* = w^*x^*
\]
\[
(w + x)^* = w^* + x^*
\]

Question: If we think of complex numbers as points on a plane, and conjugation as a function that takes a complex number to its conjugate, how does the operation of conjugation transform points on a plane?

Solution: It acts as a reflection over the $x$-axis (the real axis). (A reflection flips points over an axis of reflection.)

For $w \in \mathbb{R}$, Euler’s formula is the following:

\[
e^{iw} = \cos(w) + i\sin(w).
\]

This formula lets us use a complex exponential to represent a complex numbers.
Question: If \( w \in \mathbb{R} \), which of the following is the complex conjugate of \( e^{iw} \)?

\[
e^{-iw}, \quad \cos(w) - i \sin(w), \quad -\cos(w) + i \sin(w), \quad e^w
\] (3)

Solution: \( e^{-ix} \) and \( \cos(x) - i \sin(x) \)

If \( x = a + bi \), we denote the magnitude of \( x \) as \( |x| \), where \( |x| = \sqrt{xx^*} \).

Question: Explain why \( |e^{ix}| = 1 \) twice, using each side of Euler’s formula and the definition of magnitude. (For the left hand side, you’ll also need properties of products of exponentials, and for the right hand side, you’ll also need properties of trigonometric functions.)

Solution: First, we have \( |e^{ix}| = \sqrt{e^{ix}e^{-ix}} = \sqrt{e^{0}} = \sqrt{1} = 1 \). Then, we also have \( |e^{ix}| = |\cos(x) + i \sin(x)| = \sqrt{(\cos(x) + i \sin(x))(\cos(x) - i \sin(x))} = \sqrt{\cos^2(x) + \sin^2(x)} = \sqrt{1} = 1 \).

Question: What is a simpler expression for \( e^{i3\pi/2} \), and where does it appear as a point on the plane?

Solution: \( e^{i3\pi/2} = -i \), and it is the a point on the negative y-axis a distance 1 from the origin.

2 Vector Spaces

We will deal with vector spaces \( \mathbb{C}^d \). \( \mathbb{C}^d \) is the set of column vectors of length \( d \) (dimension \( d \)) whose elements are complex numbers. So for example

\[
\begin{pmatrix}
1 \\
i \\
 e^{2i}
\end{pmatrix} \in \mathbb{C}^3.
\] (4)

If \( x \in \mathbb{C}^d \), then the conjugate transpose of \( x \), denoted by \( x^\dagger \), is the \( d \)-dimensional row vector where the \( j \)th element of \( x^\dagger \) is the complex conjugate of the \( j \)th element of \( x \). For example, if

\[
x = \begin{pmatrix}
1 \\
i \\
 e^{2i}
\end{pmatrix} \quad \text{then} \quad x^\dagger = \begin{pmatrix}
1, & -i, & e^{-2i}
\end{pmatrix}.
\] (5)

More generally, if you have a matrix \( A \), then \( A^\dagger \) denotes the conjugate transpose of \( A \), where you take the transpose of the matrix, and then take the complex conjugate of each
element. For example
\[
\begin{pmatrix}
1, & 2, & 3 \\
-1i, & -2i, & -3i
\end{pmatrix}^\dagger =
\begin{pmatrix}
1, & 1i \\
2, & 2i \\
3, & 3i
\end{pmatrix}
\]

Given vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{C}^d \), we can take the inner product \( \mathbf{y}^\dagger \mathbf{x} \) by doing matrix multiplication between \( \mathbf{y}^\dagger \) and \( \mathbf{x} \). That is:
\[
\mathbf{y}^\dagger \mathbf{x} = \sum_{i=1}^d y_i^* x_i,
\]
where \( x_i \) is the \( i \)th element of \( \mathbf{x} \) and \( y_i \) is the \( i \)th element of \( \mathbf{y} \). For example, if \( \mathbf{x} \) is as above, and \( \mathbf{y} = \begin{pmatrix} i \\ 1+i \end{pmatrix} \), then
\[
\mathbf{y}^\dagger \mathbf{x} = (-i, 1-i, 2)^\dagger \begin{pmatrix} 1 \\ i \\ e^{2i} \end{pmatrix} = -i \times 1 + (1-i) \times i + 2 \times e^{2i} = -i + i + 1 + 2e^{2i} = 1 + e^{2i}.
\]

Question: If \( \mathbf{x} = \begin{pmatrix} 1 \\ i \end{pmatrix} \) and \( \mathbf{y} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \), what is \( \mathbf{y}^\dagger \mathbf{x} \)? What is \( \mathbf{x}^\dagger \mathbf{y} \)?
Solution: \( \mathbf{y}^\dagger \mathbf{x} = \mathbf{x}^\dagger \mathbf{y} = 0. \)

Question: If \( \mathbf{x}, \mathbf{y} \) are any vectors in \( \mathbb{C}^d \), explain why \( (\mathbf{y}^\dagger \mathbf{x})^* = \mathbf{x}^\dagger \mathbf{y} \).
Solution: Using the rules for adding and multiplying complex numbers, we have
\[
(\mathbf{y}^\dagger \mathbf{x})^* = \left( \sum_{i=1}^d y_i^* x_i \right)^* = \sum_{i=1}^d (y_i^* x_i)^* = \sum_{i=1}^d (y_i^*)^* (x_i)^* = \sum_{i=1}^d y_i x_i^* = \mathbf{x}^\dagger \mathbf{y}.
\]

Question: Show that the inner product follows the distributive property. That is, if \( \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^d \), explain why \( \mathbf{z}^\dagger (\mathbf{x} + \mathbf{y}) = \mathbf{z}^\dagger \mathbf{x} + \mathbf{z}^\dagger \mathbf{y} \).
Solution: Using the definition of inner product, we have
\[
\mathbf{z}^\dagger (\mathbf{x} + \mathbf{y}) = \sum_{i=1}^d z_i^* (x_i + y_i) = \sum_{i=1}^d z_i^* x_i + z_i^* y_i = \left( \sum_{i=1}^d z_i^* x_i \right) + \left( \sum_{j \in d} z_j^* y_j \right) = \mathbf{z}^\dagger \mathbf{x} + \mathbf{z}^\dagger \mathbf{y}.
\]

A basis for a vector space \( \mathbb{C}^d \) is a set of vectors \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_d \} \) such that for every vector
A vector \( \mathbf{x} \in \mathbb{C}^d \), there is a unique set of complex numbers \( \{a_1, \ldots, a_d\} \) such that

\[
\mathbf{x} = \sum_{j=1}^{d} a_j \mathbf{v}_j. \tag{9}
\]

If you have a set of \( d \) vectors \( \{\mathbf{v}_1, \ldots, \mathbf{v}_d\} \) each in \( \mathbb{C}^d \) such that

\[
\mathbf{v}_j^\dagger \mathbf{v}_k = \begin{cases} 
1 & \text{if } j = k \\
0 & \text{if } j \neq k,
\end{cases} \tag{10}
\]

then they form a basis for \( \mathbb{C}^d \). We call such a basis an orthonormal basis.

For example you can verify that:

\[
\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{11}
\]

from an orthonormal basis for \( \mathbb{C}^2 \) because there are two of them, and they satisfy Eq. (10). In quantum computing, we will exclusively deal with orthonormal bases.

Given a vector \( \mathbf{x} \in \mathbb{C}^d \), it is often helpful to write that vector in terms of a given orthonormal basis \( \{\mathbf{v}_1, \ldots, \mathbf{v}_d\} \). In other words, we would like to find the complex numbers \( a_j \) as in Eq. (9). When \( \{\mathbf{v}_1, \ldots, \mathbf{v}_d\} \) is an orthonormal basis, finding this decomposition is fairly straightforward. We can apply \( \mathbf{v}_i^\dagger \) to both sides of equation Eq. (9):

\[
\mathbf{v}_i^\dagger \mathbf{x} = \mathbf{v}_i^\dagger \left( \sum_{j=1}^{d} a_j \mathbf{v}_j \right)
\]

\[
= \sum_{j=1}^{d} a_j \mathbf{v}_i^\dagger \mathbf{v}_j
\]

\[
= a_i \tag{12}
\]

where the second line comes from the distributive property, and the final line comes from Eq. (10).
Question: Consider the orthonormal basis \( \{ v_1, v_2 \} \) for \( \mathbb{C}^2 \) where \( v_1 = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \) and \( v_2 = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \). Write \( x = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ i \end{array} \right) \) in this basis. (In other words, write \( x = a_1 v_1 + a_2 v_2 \) for some complex numbers \( a_1 \) and \( a_2 \).)

Solution: We have

\[
\begin{align*}
v_1^* x &= \frac{1}{2} (1 + i) \\
v_2^* x &= \frac{1}{2} (1 - i)
\end{align*}
\]

so

\[
x = \frac{1}{2} (1 + i) v_1 + \frac{1}{2} (1 - i) v_2.
\]
The tensor product has the following properties:

\[
A \otimes (B + C) = A \otimes B + A \otimes C \quad (15)
\]
\[
(B + C) \otimes A = B \otimes A + C \otimes A \quad (16)
\]
\[
(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger \quad \text{Note! the order stays the same!} \quad (17)
\]
\[
(A \otimes B) \cdot (C \otimes D) = A \cdot C \otimes B \cdot D, \quad (18)
\]

where \cdot denotes regular matrix multiplication.

**Question:** Consider the orthonormal basis \{v_1, v_2\} for \(\mathbb{C}^2\) where \(v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) and \(v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\). Show that \(\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}\) is an orthonormal basis for \(\mathbb{C}^4\).

**Solution:** Using the definition of tensor product, we have that

\[
\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\} = \left\{ \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad (19)
\]

You can check that Eq. (10) is satisfied by these vectors, and since there are 4 of them, they form an orthonormal basis for \(\mathbb{C}^4\).
Question: In this problem, we’ll show that the previous problem generalizes. Suppose we have an orthonormal basis \( \{v_1, \ldots, v_d\} \) for \( \mathbb{C}^d \), and an orthonormal basis \( \{u_1, \ldots, u_f\} \) for \( \mathbb{C}^f \). Show that the set of vectors consisting of all possible pairs of tensor products \( v_i \otimes u_j \) form an orthonormal basis, and say which space they form a basis for. (Use the tensor product properties!)

Solution: Since there are \( d \) vectors in the first set and \( f \) vectors in the second set, the number of possible pairs of vectors is \( df \). Now if we take the inner product of two of the vectors, we have, using Eq. (17)

\[
(v_i \otimes u_j)^\dagger \cdot (v_k \otimes u_l) = (v_i^\dagger \otimes u_j^\dagger) \cdot (v_k \otimes u_l).
\]  

(20)

Using Eq. (18), we have

\[
(v_i^\dagger \otimes u_j^\dagger) \cdot (v_k \otimes u_l) = (v_i^\dagger v_k) \otimes (u_j^\dagger u_l).
\]

(21)

Because \( \{v_1, \ldots, v_d\} \) and \( \{v_1, \ldots, v_f\} \) are orthonormal bases, we will get terms that are either \( 0 \otimes 0 = 0 \), \( 0 \otimes 1 = 0 \), \( 1 \otimes 0 = 0 \) or \( 1 \otimes 1 = 1 \). The only time we get 1 is when \( i = k \) and \( j = l \), which is when we take the inner product of a basis vector with itself, and otherwise we get 0. Note that these vectors are elements of \( \mathbb{C}^{df} \), so since they fulfil Eq. (10), they form an orthonormal basis for \( \mathbb{C}^{df} \).