Goals

- Analyze expectation values using indicator random variables
- Analyze comparisons of array elements

Input: Array $A$ of length $n$, no repeated elements
Output: Array with sorted elements

QuickSort(array $A$)
1. If $|A| = 1$: return $A$
2. $pivot = \text{ChoosePivot}(A)$
3. Partition($A$, pivot)
4. $A_L || A_R$

5. $\text{QuickSort}(A_L)$ (conquer)
6. $\text{QuickSort}(A_R)$

Partition
1. Move pivot to front of array
2. Maintain invariant

<table>
<thead>
<tr>
<th>val ≤ $p$</th>
<th>val &gt; $p$</th>
<th>unchecked</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$p$</td>
<td>$p$</td>
</tr>
</tbody>
</table>
Better Approach

1. Describe (generally) the sample space
   ex: \( S = \{ s : s \text{ is a sequence of possible pivot choices for our array} \} \)

2. Describe (generally) the run time random variable
   ex: \( R(s) = \# \text{ of comparisons over algorithm with pivot choices } s \)

3. Write \( R = \sum_i X_i \)
   ex: \( X_{ij}(s) = \# \text{ of times } i^{th} \text{ largest element } + j^{th} \text{ largest element are compared with pivot choices } s \)
   \( R(s) = \sum_{i,j} X_{ij}(s) \)

4. \( \mathbb{E}[R] = \sum_i \mathbb{E}[X_i] \)

   \( \mathbb{E}[X_i] = \sum_{s \in S} X_i(s) p(s) \)

   If choose \( X_i \) carefully, this is easier to calculate.
   Best \( X_i = \text{ indicator random variable: takes value 0 or 1} \)
Probability Example

Q: Suppose you have $Q$ processors and $J$ jobs. If you assign each job to a processor uniformly at random, what is the expected number of jobs processor 1 will get?

"Uniformly at random" = all equal probability

"expected" = average

A) $\sqrt{\frac{J}{Q}}$
B) $\sqrt{\frac{Q}{J}}$
C) $\frac{J}{Q}$
D) $\frac{Q}{J}$

We'll prove why!

Figuring Out Expectations:

1. Determine sample space.
   Sample Space $S = \{ s : s$ is a possible outcome $\}$

   $S =$ Set of all possible ways $J$ jobs can be assigned to $Q$ processors: $\{1, 2, \ldots, Q\}^J =$ strings of length $J$ with characters 1 to $Q$
Q: What is $|S|$?
A) $J \cdot Q$  B) $J^Q$  C) $Q^J$  D) $2^{Q+J}$

2. Define a random variable to represent quantity you care about. (Random variable = function $Y: S \to IR$)

$Y(s) =$ # of jobs assigned to processor 1 in assignment $s$.

Q: What is $Y(10, 4, 1, 3)$?
A) 0  B) 1  C) 3  D) 4
Q: What is $|S|$?
A) $J$ · $Q$  B) $J^Q$  C) $Q^J$  D) $2^{Q+J}$

Using product rule... $Q^J$

2. Define a random variable to represent quantity you care about. (Random variable = function $Y: S \rightarrow \mathbb{R}$)

$Y(s) = \#$ of jobs assigned to processor 1 in assignment $s$.

Q: What is $Y(10, 4, 1, 3)$?
A) 0  B) 1  C) 3  D) 4

↑
Only job 3 is assigned to processor 1
3. Write main random variable as a sum of indicator variables. Indicator variable is a random variable $X_A$ where

$$X_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases} \text{ for } A \subseteq S$$

$$X_j(s) = \begin{cases} 1 & \text{if job } j \text{ assigned to Processor } 1 \text{ in } i \\ 0 & \text{else} \end{cases}$$

$$Y(s) = \sum_{j=1}^{J} X_j(s)$$

4. Use Linearity of Expectation

$$E[X + Y] = E[X] + E[Y]$$

Our case:

$$E[Y] = \sum_{j=1}^{J} E[X_j]$$

$E[X]$ = expected value of variable $X$.

$\iff$ Holds even if $X_1, \ldots, X_J$ not independent random variables!
5. Calculate Expected Value:

\[ E[Y] = \sum_{s \in S} Pr[s] \cdot Y(s) \]

value of function \(X\) on input \(i\)

probability of outcome \(i\)

For indicator variable \(X_A\):

\[ E[X_A] = \sum_{s \in S} Pr[s] \cdot X_A(s) \]

\[ X_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases} \]

\[ = \sum_{s \in E} Pr(s) \cdot 1 + \sum_{s \notin A} Pr(s) \cdot 0 \]

\[ = \sum_{s \in E} Pr(s) \]

Definition of Probability

of event \(A\).

Want to choose indicator variables where figuring out this probability is easy.

\[ E[Y] = \sum_{j=1}^{J} E[X_j] \]

\[ = \sum_{j=1}^{J} Pr[\text{\(j^{th}\) job assigned to 1st processor}] \]

\[ = \sum_{j=1}^{J} \frac{1}{P} = \frac{J}{P} \]
Back to QuickSort:

\[ R = \sum_{i,j} X_{i,j} \]

# comparisons b/t i\textsuperscript{th} largest + j\textsuperscript{th} largest array elements on element of sample space

Let \( z_i \) = \( i \textsuperscript{th} \) largest element of A

\[
\begin{array}{c}
5 \\
2 \\
1 \\
\end{array} \quad \begin{array}{c}
2 \\
1 \\
3 \\
\end{array}
\]

**Story of \( z_i, z_j \):**

- As long as pivot = \( z_k \) with \( k > i, j \) or \( k < i, j \), \( z_i, z_j \) get put together into same subarray for recursion. (No comparisons)

- Something interesting happens when pivot is \( z_k \) with \( i \leq k \leq j \)

Because \( X_{i,j} \) is 0 or 1, this means \( X_{i,j} \) is an indicator random variable!

\[ \begin{array}{c}
z_i, z_{i+1}, \ldots, z_{j-1} \\
z_j \\
\end{array} \]

\( K = i \text{ or } j \)

- 1 comparison

\( i < k < j \)

- 0 comparisons

(No further comparisons because pivot is not included in recursive calls)

(No further comparisons because \( z_i \) and \( z_j \) separated: \( z_i \) in \( A_L \), \( z_j \) in \( A_R \).)