Goals

- Prove correctness of greedy scheduling
- Prove correctness of loops using loop invariants

Recall

Scheduling: $n$ jobs

- $t_i$: time to run job $i$
- $w_i$: weight (importance) of job $i$
- $C_i$: time to complete job $i$

Want to minimize $A = \sum w_i C_i$

We’ve tried several greedy algorithms, we think

$$f(w, t) = \frac{w}{t}$$

is optimal (order by largest $f$)
Thm: Greedy algorithm with \( f = \frac{w_i}{t_i} \) is optimal for objective function \( \sum w_i \cdot c_i \).

**Pf:** \textsc{Exchange} argument \( \text{(Proof by Contradiction)} \)

Assume \( \frac{w_i}{t_i} \) are distinct \( \forall i \in \{1, 2, \ldots, n\} \)

WLOG, relabel so \( \frac{w_1}{t_1} > \frac{w_2}{t_2} > \ldots > \frac{w_n}{t_n} \)

Let \( \sigma \) be ordering using greedy, so \( \sigma = (1, 2, 3, \ldots, n) \)

For contradiction assume that the optimal ordering \( \sigma^* \) is not the greedy ordering.

Note: \( \exists k, j \text{ s.t. } \frac{w_j}{t_j} > \frac{w_k}{t_k} \text{, but } j \text{ is immediately after } k \text{ in } \sigma^* \text{ ordering.} \)

(Otherwise, \( \sigma^* = \sigma \))

Let's create a new ordering \( \sigma^*' \) that is same as \( \sigma^* \), but with \( k, j \) positions switched.

If \( \sigma^* \) has objective value \( A_{\sigma^*} \), and \( \sigma^*' \) has objective value \( A_{\sigma^*'} \), what is \( A_{\sigma^*} - A_{\sigma^*'} \)?
A scheduling problem is discussed with the time to complete the first set of jobs denoted as $T$. The expressions for $A_{\sigma^*}$ and $A_{\sigma'^*}$ are given:

$$A_{\sigma^*} = \sum w_i C_i + w_k (T + t_k) + w_j (T + t_k + t_j) + \sum w_r c_r$$

$$A_{\sigma'^*} = \sum w_i C_i + w_j (T + t_j) + w_k (T + t_j + t_k) + \sum w_r c_r$$

The difference $A_{\sigma^*} - A_{\sigma'^*} = w_j t_k - w_k t_j$ is highlighted. It is shown that if $w_j > w_k$ and $t_j > t_k$, then $w_j t_k > w_k t_j$, leading to $A_{\sigma'^*} < A_{\sigma^*}$, which is a contradiction because $\sigma^*$ is optimal.

Thus, our assumption that $\sigma$ was not optimal was incorrect and $\sigma$ is an optimal schedule.

The question at the bottom asks, "What is the runtime of the greedy scheduling algorithm?"

A) $O(1)$  B) $O(n)$  C) $O(n \log n)$  D) $O(n^2)$
Q: What is the runtime of the greedy scheduling alg.? 
A) \(O(1)\)  B) \(O(n)\)  C) \(O(n\log n)\)  D) \(O(n^2)\)