Max-Weight Independent Set Problem (MWISP)

Input: Graph \( (V, E) \) and weight function \( w : V \rightarrow \mathbb{R}^+ \)

Output: \( S \subseteq V \) s.t. if \( (v_i, v_j) \in E \), \( v_i, v_j \) can't both be in \( S \).

\( S \) maximizes \( w(S) = \sum_{v_i \in S} w(v_i) \) \( \quad \Rightarrow \) max weight

This set is Max Weight Ind. Set (MWIS)

MWIS

1. Think about form of Soln
2. Related soln to smaller soln

MWIS on \( G_n \) is

i) MWIS on \( G_{n-1} \)

OR

ii) MWIS on \( G_{n-2} + v_n \)

\{ Take max of these two options! \}
3. Create a recurrence for objective function value:

\[ A[i] = \max \{ A[i-1], A[i-2] + w(v_i) \} \]

\[ \overset{\text{case (i)}}{\uparrow} \quad \overset{\text{(ii)}}{\uparrow} \]

Q: What is \( A[0], A[1] \)?

\[ \begin{align*}
A[0] &= 0 \\
A[1] &= 1 \\
0 &\leq w(v_i)
\end{align*} \]

Fill in \( A \) using loop:

\[
\begin{aligned}
A[0] &= 0 \\
A[1] &= 1 \\
\text{for } i = 2 \text{ to } n: \\
\quad A[i] &= \max \{ A[i-1], A[i-2] + w(v_i) \}
\end{aligned}
\]

\[ \Rightarrow \quad \text{Run Time: } O(n) \]

Correctness: Induction
EX:

\[ v_1 \quad v_2 \quad v_5 \quad v_4 \quad v_3 \]

5 7 4 4 1

\[
\begin{array}{ccccccc}
A & 0 & 5 & 7 & 9 & 11 & 11 \\
0 & 1 & 2 & 3 & 4 & 5 & S = \emptyset
\end{array}
\]


Q: Write pseudo code to get \( S \) given \( A \)

\[
S = \emptyset \\
i = n \\
\text{while } i > 0 \\
\quad \text{if } A[i] = A[i-2] + V_n \\
\quad \quad S = S + i \\
\quad \quad i = i - 2 \\
\quad \text{else } i = i - 1 \\
\text{end while} \\
\{ O(n) \text{ runtime!} \} 
\]
Indicator Random Variables Again

Interview n ppl for job, if interviewee is better than current, immediately hire.

How many hires depends on ranking of people

\[ \{3, 4, 1, 5, 2\} \]

1st person better \uparrow \text{hired}

Suppose ranking order is random. How many hires?

Sample space \( \Omega \): all possible ranking orders (ways to order numbers 1, 2, \ldots, n) size = \( n! \)

\[ H(s) = \# \text{ of hires given ranking } s \in \Omega \] (random variable)

\[ = \sum X_i(s) \]

indicator random variables

"How can I break big random variable into a sum of random variables that take value 1 or 0?"

\[ X_i(s) = \begin{cases} 1 & \text{if person } i \text{ is hired in ordering } s \\ 0 & \text{otherwise} \end{cases} \]
\[ H(s) = \sum_{i=1}^{n} X_i(s) \]

\[ E[H] = E\left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} E[X_i] \]

\[ = \sum_{i=1}^{n} \Pr(\text{event associated with } X_i \text{ occurs}) \]

\[ = \sum_{i=1}^{n} \Pr(\text{Person } i \text{ is best so far}) \]

Consider the first \( i \) elements. We include all permutations of these elements. Want \( i \) to be largest. \( \Rightarrow \) Occurs w/prob \( \frac{1}{i} \)

\[ E[H] = \sum_{i=1}^{n} \frac{1}{i} \leq \ln(n) + 1. \]