1. Prove that the algorithm laid out in pseudocode in algorithm 1 is correct.

**Input**: Two integers $a$ and $b$, given as arrays of length $n$, where
\[ a = [a_{n-1}, \ldots, a_1, a_0], \quad b = [b_{n-1}, \ldots, b_1, b_0]. \]

**Output**: $a \times b = \left( \sum_{i=0}^{n-1} a_i 10^i \right) \left( \sum_{j=0}^{n-1} b_j 10^j \right)$ as an integer

1 if $n == 1$ then
  2 return $a \times b$; // Base case when both numbers are 1-digit numbers
else
  // Divide input into halves, and pad with zeros if necessary:
  4 $h = \lfloor n/2 \rfloor$;
  5 $a^1 = [a_{n-1}, \ldots, a_h]$;
  6 $b^1 = [b_{n-1}, \ldots, b_h]$;
  7 if $n - h \neq h$ then
    8 $a^0 = [0, a_{h-1}, \ldots, a_0]$;
    9 $b^0 = [0, b_{h-1}, \ldots, b_0]$;
  else
    11 $a^0 = [a_{h-1}, \ldots, a_0]$;
    12 $b^0 = [b_{h-1}, \ldots, b_0]$;
  end
  // Conquer!
  14 return $10^h \times \text{RecMultiplication}(a^1, b^1, n - h) + \text{RecMultiplication}(a^0, b^0, n - h)$;
end

Algorithm 1: \text{RecMultiplication}(a, b, n)

**Solution**

We will prove using strong induction on $n$, the length of $a$ and $b$, that \text{RecMultiplication} correctly outputs the product of $a$ and $b$.

For the base case, if $n = 0$, the algorithm does nothing, which is correct, since $a$ and $b$ have length 0.

Now for the inductive step. For strong induction, we assume the algorithm outputs the correct result when the length of the input is $k$, for all $k$ such that $n > k \geq 0$. We
will prove the algorithm outputs the correct value on inputs of size $n$. Since $n \geq 0$, the algorithm enters the recursive case. Note that each of the recursive calls in line 14 involves a multiplication of two numbers with $n-h$ digits (thanks to our padding with zeros step), where $h = \lfloor n/2 \rfloor \geq 1$.

Therefore, the algorithm returns

$$\begin{align*}
n & = 10^{2h} \left( \sum_{i=0}^{n-h-1} a_i 10^i \right) \left( \sum_{j=0}^{n-h-1} b_j 10^j \right) + \left( \sum_{i=0}^{n-h-1} a_i 10^i \right) \left( \sum_{j=0}^{n-h-1} b_j 10^j \right) \\
& \quad + 10^h \left( \sum_{i=0}^{n-h-1} a_i 10^i \right) \left( \sum_{j=0}^{n-h-1} b_j 10^j \right) + 10^h \left( \sum_{i=0}^{n-h-1} a_i 10^i \right) \left( \sum_{j=0}^{n-h-1} b_j 10^j \right) \\
& = (10^h \sum_{i=0}^{n-h-1} a_i 10^i + \sum_{i=0}^{n-h-1} a_i 10^i) \left(10^h \sum_{j=0}^{n-h-1} b_j 10^j + \sum_{j=0}^{n-h-1} b_j 10^j\right) \\
& = a \times b.
\end{align*}$$

Thus, by strong induction, the algorithm is correct.