CS302 - Problem Set 0

1. Prove that the algorithm laid out in psuedocode in algorithm 1 is correct.

Input : Two integers a and b, given as arrays of length n, where $a = [a_{n-1}, \ldots, a_1, a_0]$, $b = [b_{n-1}, \ldots, b_1, b_0]$.

Output: $a \times b = \left(\sum_{i=0}^{n-1} a_i 10^i\right)\left(\sum_{j=0}^{n-1} b_j 10^j\right)$ as an integer

if $n == 1$ **then**
  return $a * b$;`// Base case when both numbers are 1-digit numbers`
else
  `// Divide input into halves, and pad with zeros if necessary:`
  $h = \lfloor n/2 \rfloor$;
  $a^1 = [a_{n-1}, \ldots, a_h]$
  $b^1 = [b_{n-1}, \ldots, b_h]$
  if $n - h \neq h$ **then**
    $a^0 = [0, a_{h-1}, \ldots, a_0]$
    $b^0 = [0, b_{h-1}, \ldots, b_0]$
  else
    $a^0 = [a_{h-1}, \ldots, a_0]$
    $b^0 = [b_{h-1}, \ldots, b_0]$
  end
  `// Conquer!`
  return $10^{2h}\times$`RecMultiplication`$(a^1, b^1, n-h)$ $+$`RecMultiplication`$(a^0, b^0, n-h)$ $+10^h\times$`RecMultiplication`$(a^0, b^1, n-h)$ $+10^h\times$`RecMultiplication`$(a^1, b^0, n-h)$
end

Algorithm 1: `RecMultiplication`(a, b, n)

Solution

We will prove using strong induction on n, the length of a and b, that `RecMultiplication` correctly outputs the product of a and b.

For the base case, if $n = 0$, the algorithm does nothing, which is correct, since a and b have length 0.

Now for the inductive step. For strong induction, we assume the algorithm outputs the correct result when the length of the input is k, for all k such that $n > k \geq 0$. We will prove the algorithm outputs the correct value on inputs of size n. Since $n \geq 0$, the algorithm enters the recurisive case. Note that each of the recursive calls in line 14 involves a multiplication of two numbers with $n - h$ digits (thanks to our padding with zeros step), where $h = \lfloor n/2 \rfloor \geq 1$.

Therefore, the algorithm returns

$$\begin{aligned}
&10^{2h} \left(\sum_{i=0}^{n-h-1} a_i^1 10^i \right) \left(\sum_{j=0}^{n-h-1} b_j^1 10^j \right) + \left(\sum_{i=0}^{n-h-1} a_i^0 10^i \right) \left(\sum_{j=0}^{n-h-1} b_j^0 10^j \right) \\
&+ 10^h \left(\sum_{i=0}^{n-h-1} a_i^0 10^i \right) \left(\sum_{j=0}^{n-h-1} b_j 10^j \right) + 10^h \left(\sum_{i=0}^{n-h-1} a_i^1 10^i \right) \left(\sum_{j=0}^{n-h-1} b_j^0 10^j \right) \\
=& \left(10^h \sum_{i=0}^{n-h-1} a_i^1 10^i + \sum_{i=0}^{n-h-1} a_i^0 10^i \right) \left(10^h \sum_{j=0}^{n-h-1} b_j^1 10^j + \sum_{j=0}^{n-h-1} b_j^0 10^j \right) \\
=& a \times b. \qquad (1)
\end{aligned}$$

Thus, by strong induction, the algorithm is correct.