CS302 - Problem Set 0

1. Prove that the algorithm is correct.

**Solution**  Let $P(n)$ be the predicate that RecMultiplication correctly multiplies two $n$-digit numbers, for $n \geq 1$. We prove correctness by strong induction on $n$.

For the base case, if $n = 1$, the algorithm triggers the base case and returns $a \times b$, so $P(n)$ is true.

Now for the inductive step. For strong induction, we assume that $P(k)$ is true for all $k$ such that $n > k \geq 1$. Using this assumption, we will prove $P(n)$ is true. Since $n > 1$, the recursive case of the algorithm triggers. Note that each of the recursive calls in line 14 involves a multiplication of two numbers with $n - h$ digits (thanks to our padding with zero steps), where $h = \lfloor n/2 \rfloor \geq 1$. Since $n - h < n$, we may assume the algorithm returns the correct output on these inputs. That is, it returns the product of the two inputs.

Therefore, the algorithm returns

$$
10^{2h} \left( \sum_{i=0}^{n-h-1} a_i 10^i \right) \left( \sum_{j=0}^{n-h-1} b_j 10^j \right) + \left( \sum_{i=0}^{n-h-1} a_i 0 10^i \right) \left( \sum_{j=0}^{n-h-1} b_j 0 10^j \right)
+ 10^h \left( \sum_{i=0}^{n-h-1} a_i 0 10^i \right) \left( \sum_{j=0}^{n-h-1} b_j 10^j \right) + 10^h \left( \sum_{i=0}^{n-h-1} a_i 1 10^i \right) \left( \sum_{j=0}^{n-h-1} b_j 0 10^j \right)
= \left( 10^h \sum_{i=0}^{n-h-1} a_i 10^i + \sum_{i=0}^{n-h-1} a_i 0 10^i \right) \left( 10^h \sum_{j=0}^{n-h-1} b_j 10^j + \sum_{j=0}^{n-h-1} b_j 0 10^j \right)
=a \times b.
$$

Thus, by strong induction, the algorithm is correct for all $n \geq 1$. 

\(1\)