Learning Goals
- Properly use multiple base cases in a strong inductive proof.
- Describe graphs using math language
- Describe some real world graph applications.

\( F(n) \)
1. If \( n \leq 1 \): return \( n \)
2. return \( 5 \cdot F(n-1) - 6 \cdot F(n-2) \)

Q: Prove this algorithm returns \( 3^n - 2^n \) for all \( n \geq 0 \).

Let \( P(n) \) be the predicate \( F(n) \) returns \( 3^n - 2^n \). We will prove \( P(n) \) is true for all \( n \geq 0 \), using strong induction.

**Base cases**: We will show \( P(0) \) and \( P(1) \). When the input is 0, we return 0. Since \( 3^0 - 2^0 = 1 - 1 = 0 \), this is correct. When the input is 1, we return 1. Since \( 3^1 - 2^1 = 3 - 2 = 1 \), this is correct.

**Inductive step**: Let \( k \geq 1 \). Assume \( P(j) \) is true for all \( j \) such that \( 0 \leq j \leq k \). We will prove \( P(k+1) \).

We want \( k+1 \) to be larger than base cases, so choose \( k \) to be larger than or equal to largest base case.

We need to prove \( P(0) \) and \( P(1) \). Otherwise when trying to prove \( P(2) \), look at \( f(2-1) = f(1) \) and \( f(2-2) = f(0) \), need to assume these output correctly.
Let $P(n)$ be the predicate $F(n)$ returns $3^n - 2^n$. We will prove $P(n)$ is true for all $n \geq 0$, using strong induction.

**Base case:** There are two base cases: when the input is 0, we return 0. Since $3^0 - 2^0 = 1 - 1 = 0$, this is correct. When the input is 1, we return 1. Since $3^1 - 2^1 = 3 - 2 = 1$, this is correct.

**Inductive step:** We assume $P(j)$ is true for all $j \leq k$. Now consider input $k+1$. Now $k+1 \geq 2$, so we return $5F(k) - 6F(k-1)$. Since $k \geq 1$, $F(k)$ returns $3^k - 2^k$ by inductive assumption. Since $k-1 \geq 0$, and $F(0)$ is correct by the base case, and larger values are true by inductive assumption, $F(k-1)$ returns $3^{k-1} - 2^{k-1}$. Thus the function returns

$$5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1})$$
$$= 5 \cdot 3^k - 5 \cdot 2^k - 2 \cdot 3^{k-1} + 3 \cdot 2^{k-1}$$
$$= 3^{k+1} - 2^{k+1}$$

as desired.

Therefore, by strong induction, $F(n)$ is correct.
Graphs

\[ G = (V, E) \]

- Use parentheses to denote ordered set

\[ V = \text{set of vertices} \]

*ex: \( V = \{a, b, c, d, e\} \)

\[ E = \text{set of edges} \]

*ex: \( E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{d, e\}, \{b, e\}, \{c, e\}\} \)

- Each edge is a set consisting of 2 vertex elements.

**Draw this Graph:**

- a
- b
- c
- d
- e
Graphs:

- **Vertices**, nodes
- **Edges**

**Natural questions:**
- Which vertex has the largest degree? (degree of $v \in V$ is $\sum_{u \in V, u \neq v} 1$)
- Are two nodes connected? <map
- What is the shortest path from one node to another?
- What are the fewest edges one would need to remove to separate two nodes? <cyber attack, rail way attack

- **Is the graph $G=(V,E)$ complete?** (complete: $\forall u,v \in V, u \neq v \rightarrow \{u,v\} \in E$)
  - There is an edge b/w each pair of vertices
  - $K_4$ - complete graph on 4 vertices

- **Is graph $G=(V,E)$ bipartite?**
  - Bipartite: $\exists S,T \subseteq V: S \cup T = V \setminus \{S \cap T\}$
  - $\forall \{a,b\} \in E: (a \in S \land b \in T) \lor (a \in T \land b \in S)$