Goals

- Prove (not) equivalence relation
- Identify equivalence classes
- Analyze runtime of For-loops

Reflections

- This pset is challenging
  - Identify proof strategy
    - If stuck, try different approach
    - Will develop intuition with practice
  - Combining functions/counting/graphs
    - Make sure basics are solid

Equivalence Relations

Why do we care? They allow us to define equivalence classes.

\[ R \text{ is an equivalence relation on } S \ (\text{i.e. } R \subseteq S \times S) \]

\[ (a, b) \in R \iff a \text{ and } b \text{ are in the same equivalence class} \]
Q: Let $S = \{0, 1, 2, 3, 4\}$. Let $R \subseteq S \times S$ be the equivalence relation:

$$R = \{(0,0), (0,1), (1,0), (1,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$$

What are equivalence classes?

A) $\{0, 1, 3, 4\}$

B) $\{0, 1\}, \{3, 4\}$

C) $\{0, 1\}, \{2\}, \{3, 4\}$

D) $\{(0,0), (1,1)\}, \{(2,2)\}, \{(3,3), (4,4)\}$
Q: Let $S = \{0, 1, 2, 3, 4\}$. Let $R \subseteq S \times S$ be the equivalence relation:

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R = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}
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D) $\{(0, 0), (1, 1)\}, \{(2, 2)\}, \{(3, 3), (4, 4)\}$

Divide

```
   0  1  2

  3  4
```

Each is an equivalence class.

Equivalence classes are subsets of $S$. 
To prove $R$ is equivalence relation:
• 3 mini proofs (usually 1 line each) or explanations
  - Symmetric, Reflexive, Transitive

To prove $R$ is not an equivalence relation
• Show 1 property doesn’t hold
• Properties have form $\forall a \in S$...
⇒ Need counterexample to show $\not\forall a \in S$...
Input: Adjacency Matrix $A$ for $G = (V, E)$, $G$ unweighted
Output: ??

1. $S = 0$
2. for $i = 1$ to $|V|$
   i. for $j = 1$ to $i$
   j. $S += A[i, j]$
3. Return $S$

How many operations?
- Use $\Sigma$ for loops
- Use 1 for $O(1)$ operations

\[
\text{# operations} = 1 + \sum_{i=1}^{\lfloor \frac{|V|}{2} \rfloor} \left( \sum_{j=1}^{i} 1 \right) \quad [\text{work done inside } i^{th} \text{ loop iteration}]
\]

Write your expression from outer loops to inner loop
Evaluate from the inside out:

\[
\# \text{ operations} = 1 + \sum_{i=1}^{\|V\|} \left( \sum_{j=1}^{i} 1 \right)
\]

\[
= 1 + \sum_{i=1}^{\|V\|} i
\]

\[
= 1 + \left( 1 + 2 + 3 + \ldots + \|V\| \right)
\]

\[
= 1 + \frac{(\|V\|+1)\|V\|}{2}
\]

\[
= \mathcal{O}(\|V\|^2)
\]

"Detailed Calculation"
Input: Adjacency Matrix A for \( G = (V, E) \), \( G \) unweighted
Output: ??

1. \( S = 0 \)
2. for \( i = 1 \) to \( |V| \):
   j
3. for \( j = 1 \) to \( i \):
   i
   \( S += A(i, j) \)
4. \( S += A(i, j) \)
5. Return \( S \)

This time doesn't over count!
Returns \( |E| \)

How many operations?

Worst-case Calculation:

line 2: Repeats \( |V| \) times:

\[ \rightarrow \text{line 3: Worst case repeats } |V| \text{ times because } i \leq |V| \]

\[ \rightarrow \text{Does constant work} \]

Nested loops: \( O(|V| \times |V|) \)

+ Constant

\( = O(|V|^2) \)

Big-O is upper bound!