Please read the sections of the syllabus on problem sets and honor code before starting this homework.

1. **Inductive Proofs**

   (a) **[11 points]** Prove using induction that for \( n \geq 0 \), \( 7^n - 2^n \) is divisible by 5. (An integer \( m \) is divisible by an integer \( r \) if \( m = r \cdot g \), where \( g \) is some other integer.)

   (b) **[11 points]** Prove using induction that \( 2^n > n^2 \) whenever \( n \) is an integer, and \( n \geq 5 \).

   (c) **[11 points]** Prove that \( 1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6 \) for any integer \( n \) such that \( n \geq 1 \).

   (d) **[11 points]** Finish the following proof that Algorithm 1 correctly multiplies an integer \( n \geq 0 \) and an integer \( b \).

   Algorithm 1: Mult\((n,b)\)

   ```
   Input  : Non-negative integer \( n \), and integer \( b \)
   Output: \( n \times b \)
   /* Base Case
   1 if \( n == 0 \) then
   2     return 0;
   3 else
   4     // Recursive step
   5     return \( b + \text{Mult}(n-1,b) \);
   5 end
   */
   ```

   **Proof:** Let \( P(n) \) be the predicate: Mult\((n,b)\) correctly outputs the product of \( n \) and \( b \). We will prove using induction that \( P(n) \) is true for all \( n \geq 0 \).

   For the base case, let \( n = 0 \). In this case, we see the `If` statement is true at line 1, and so the algorithm returns 0. This is precisely what we want, since \( 0 \times b = 0 \) for any integer \( b \), so the algorithm is correct and \( P(0) \) is true.

   For the inductive step...

2. **[3 points each]** For each of the following sentences, decide whether it is a statement, predicate, or neither, and explain why

   (a) Call me Ishmael.

   (b) The world is supported on the back of a giant tortoise.

   (c) \( x \) is a multiple of 7.
(d) The next sentence is true.
(e) The preceding sentence is false.
(f) The set \( \mathbb{Z} \) contains an infinite number of elements.

3. [3 points each]
There are many ways to represent the logical implication \((P \rightarrow Q)\) in English. To make proofs more interesting to read, we often take advantage of these different ways of phrasing the same underlying mathematical statement. In the following, I will ask you to rewrite sentences in the form \( p \rightarrow q \). For example, “I get a brain freeze if I eat ice cream” should be rewritten “I eat ice cream \( \rightarrow \) I get a brain freeze.” Normally there are two clear possibilities: \( p \rightarrow q \) or \( q \rightarrow p \) and only one of them makes sense. If you are having trouble, check out p. 43 of Book of Proof, or problem 5 in Chapter 0 of DMOI (which has solutions).

(a) I open my umbrella whenever it rains.
(b) I miss class only if I am unwell.
(c) You can’t invent unless you are curious and knowledgeable.

4. [6 points] Prove using a truth table that \(((A \lor B)) \land (A \rightarrow C) \land (B \rightarrow C)) \rightarrow C\) is true.
(Note \( P \lor Q \lor R \) is only true when all of the predicates are true, and is false otherwise.) This statement is also known as “proof by cases.” Please explain why. (Hint: the two cases are related \( A \) and \( B \), and we want to say something about \( C \).)

5. How long did you spend on this homework?