Name ____________________

CS200 Worksheet - Review

1. For each of the following, prove true or prove false:

   (a) $\lceil 2x \rceil = O(x)$

   (b) $2^x + 17 = O(3^x)$

   (c) $x \log(x) = O(x)$

   Solution

   (a) Note that for all x, we have $\lceil 2x \rceil \leq 2x + 1$, because the ceiling function at most rounds up to the nearest integer. Then for all $x \geq 1$, we have $2x+1 \leq 2x+x = 3x$. Thus with $k = 1$ and $C = 3$, $\lceil 2x \rceil = O(x)$.

   (b) Note that $3^3 = 27$, so for all $x \geq 3$, we have $17 < 3^x$. Also, for all $x \geq 1$, $2^x \leq 3^x$. Thus $2^x + 17 \leq 3^x + 3^x = 2 \times 3^x$. Thus with $k = 3$ and $C = 2$, $2^x + 17 = O(3^x)$.

   (c) $x \log_2(x)$ is not $O(x)$. For contradiction, suppose there exist k and C such that for all $x \geq k$, we have $x \log_2(x) \leq Cx$. For $x \geq 0$, we can divide both sides by x to get $\log_2(x) \leq C$. But for any constant C, this inquality is false for $x \geq 2^C$; in other words, it is not true for all x larger than k, a contradiction.

2. Consider a game in which there are two players and two piles of tokens, and the piles both start out with the same number of tokens. The players alternate turns. On each player's turn, they choose a pile, and can remove as many tokens as they want from that pile. The player who removes the last token remaining wins. Prove that the player who goes second can always win.

   Solution Let $P(n)$ be the statement that the second player will win if there are initially n tokens in each piles. We will prove using strong induction that $P(n)$ is true for all $n \geq 1$.

   Base Case: If $n = 1$, then there is one token in each pile, so the first player must remove one token from one of the piles, which leaves the second player to remove the final token and win.

   Inductive step: Suppose $P(r)$ is true for all r such that $1 \leq r \leq k$. We will prove that $P(k+1)$ is true. There are two cases. In the first case, the first player removes all $k+1$ tokens from one of the piles. Then the second player can remove all $k+1$ tokens from the other pile and win. In the second case, the first player removes t tokens, for $1 \leq t < k+1$, from one of the piles, this means there are $k+1-t$ tokens in one of the piles. Then the second player can remove t tokens from the other pile.

Now it is the first player's turn, and there are $k+1-t$ tokens in both piles, where $1 \leq k+1-t < k+1$. This is like starting a new game with $k+1-t$ tokens, so by our inductive assumption, $P(k+1-t)$ is true, so the second player will win the game.

Thus by strong induction $P(n)$ is true for all $n \geq 1$.

3. Prove using a proof by contradiction that if $3n+2$ is odd, then n is odd. (Note that you can prove this statement using other approaches, but for practice, use contradiction.)

   Solution For contradiction, suppose $3n+2$ is odd, and n is even. Then $n = 2b$ for $b \in \mathbb{Z}$. Plugging in, this means $3n+2 = 6b+2 = 2(3b+1)$. Since $3b+1$ is an integer, $3n+2$ is even, which contradicts our assumption that $3n+2$ is odd. Thus the statement is true.

4. Prove $\log_2(3)$ is irrational.

   Solution Suppose for contradiction, that $\log_2(3) = a/b$, where $a, b \in \mathbb{Z}$. Furthermore, we can choose both a and b to be positive, since $\log_2(3)$ is positive. Then we can raise both sides to the power of 2 to get $3 = 2^{a/b}$. Now if we raise both sides to the power of b, we get $3^b = 2^a$. But since a and b are positive, notice that the right hand side is even, but the left hand side is odd, a contradiction. Therefore, $\log_2(3)$ is irrational.