S.KIMMEL

Q: Consider strings of $\{1,2,4\}^n$, where

$Pr(k^{th} \text{ digit is } 1) = 1/k$

$Pr(k^{th} \text{ digit is } 2) = (1-1/k)/2$

$Pr(k^{th} \text{ digit is } 4) = (1-1/k)/2$

1. See average. Need sample space, random variable

$\{1,2,4\}^n$ $\qquad$ $X(i) = $ sum of digits in i

2. Write X as a weighted sum of indicator random variables (can do in several steps)

$$X = 1 \cdot \underbrace{X_1}_{\#\,1\text{'s}} + 2 \cdot \underbrace{X_2}_{\#\,2\text{'s}} + 4 \cdot \underbrace{X_4}_{\#\,4\text{'s}}$$

$$X_1 = \sum_{k=1}^{n} X_{1,k} \qquad X_{1,k} = \begin{cases} 1 & \text{if } k^{th} \text{ position is } 1 \\ 0 & \text{otherwise} \end{cases}$$

$$X_2 = \sum_{k=1}^{n} X_{2,k} \qquad X_{2,k} = \begin{cases} 1 & \text{if } k^{th} \text{ position is } 2 \\ 0 & \text{otherwise} \end{cases}$$

$$X_4 = \sum_{k=1}^{n} X_{4,k} \qquad X_{4,k} = \begin{cases} 1 & \text{if } k^{th} \text{ position is } 4 \\ 0 & \text{otherwise} \end{cases}$$

$$X = \sum_{k=1}^{n} X_{1,k} + 2\sum_{k=1}^{n} X_{2,k} + 4\sum_{k=1}^{n} X_{4,k}$$

3. Use linearity of expectation:

$$\mathbb{E}[X] = \sum_{k=1}^{n} \mathbb{E}[X_{1,k}] + 2\sum_{k=1}^{n} \mathbb{E}[X_{2,k}] + 4\sum_{k=1}^{n} \mathbb{E}[X_{4,k}]$$

S.KIMMEL

5. Use $\mathbb{E}[X_E] = \Pr(E)$ for indicator random variables.

$$\mathbb{E}[X] = \sum_{k=1}^{n} \underbrace{\mathbb{E}[X_{1,k}]}_{\frac{1}{k}} + \sum_{k=1}^{n} 2\underbrace{\mathbb{E}[X_{2,k}]}_{(1-1/k)/2} + \sum_{k=1}^{n} \underbrace{\mathbb{E}[X_{4,k}]}_{(1-1/k)/2} 4$$

$$= \sum_{k=1}^{n} \frac{1}{k} + 1 - \frac{1}{k} + 2 - 2/k$$

$$= \sum_{k=1}^{n} 3 - \frac{2}{k} = 3n - O(\log n)$$

$$\sum_{k=1}^{n} \frac{1}{k} \approx \ln(n)$$