def: Given a sample space $S$, a random variable $X$ is a function $X: S \to \mathbb{R}$.

ex: Let $S$ be the sample space consisting of all possible outcomes of 4 coin tosses. Let $X$ be the number of heads that occur.

Q: What is $X(T,H,H,H)$? What is $X(T,T,H,T)$?

A: 1, 3  B: 2, 2  C: 3, 1  D: 4, 4

def: The expected or average value of a random variable $X$ is

$$E[X] = \sum_{i \in S} \Pr(i) X(i).$$

Q: From previous example, what is $E[X]$? (The average number of heads in 4 coin flips.)

A) 1  B) 2  C) 2.5  D) 4
I’m guessing you didn’t do the following:

\[ E[X] = \sum_{i \in S} \Pr(i)X(i) \quad (2^4 \text{ elements of sample space!}) \]

\[ E[X] = \sum_{i \in S: X(i)=0} \Pr(i) \cdot 0 + \sum_{i \in S: X(i)=1} \Pr(i) + \sum_{i \in S: X(i)=2} \Pr(i) \cdot 2 \]

\[ + \sum_{i \in S: X(i)=3} \Pr(i) \cdot 3 + \sum_{i \in S: X(i)=4} \Pr(i) \cdot 4 \]

... good practice to finish on your own!

\[ \Pr(i) = \frac{1}{2^4} = \frac{1}{16} \text{ in all cases.} \]

\[ \left| \{i \in S: X(i)=0\} \right| = 1 \quad \left| \{i \in S: X(i)=2\} \right| = \binom{4}{2} = 6 \]

\[ \left| \{i \in S: X(i)=1\} \right| = \binom{4}{1} = 4 \quad \left| \{i \in S: X(i)=3\} \right| = \binom{4}{3} = 4 \]

\[ \left| \{i \in S: X(i)=4\} \right| = 1 \]

\[ E[X] = \frac{1}{16} \left( 1 \cdot 0 + 4 \cdot 1 + 6 \cdot 2 + 4 \cdot 3 + 1 \cdot 4 \right) \]

\[ = \frac{1}{16} (32) = 2 \]

Instead you used indicator random variables + linearity of expectation (without knowing!)
Indicator Random Variable:

An **indicator random variable** $X$ is a random variable such that $X : S \rightarrow \{0, 1\}$.

An indicator random variable is associated with an event $E \subseteq S$:

$$E = \{ i \in S : X(i) = 1 \}$$

Normally write as $X_E$ where

$$X_E(s) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E[X_E] = \sum_{i \in S} \Pr(i) \cdot X_E(i)$$

$$= \sum_{i \in S : X_E(i) = 0} \Pr(i) \cdot 0 + \sum_{i \in S : X_E(i) = 1} \Pr(i)$$

$$= \sum_{i \in E} \Pr(i) = \Pr(E)$$

$$E[X_E] = \Pr(E)$$
**Linearity of Expectation**

Let $Y_1, Y_2, \ldots, Y_n$ be random variables on a sample space $S$. Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$. Let $Y$ be a random variable s.t.

\[ \forall i \in S, \quad Y(i) = \sum_{k=1}^{n} a_k Y_k(i) \]

Then

\[ E(Y) = E\left( \sum_{k=1}^{n} a_k Y_k \right) = \sum_{k=1}^{n} a_k E(Y_k) \]

**Ex:** Let $X_k$ be the indicator random variable that takes value 1 if $k$-th coin flip is Heads. Let $X = \#$ of heads in 4 coin tosses

Then

\[ X = \sum_{k=1}^{4} X_k \]

\[ \text{e.g. } X(H,T,T,H) = X_1(H,T,T,H) + X_2(H,T,T,H) + X_3(H,T,T,H) + X_4(H,T,T,H) = 2 \]

\[ E[X] = \sum_{k=1}^{4} E[X_k] = \sum_{k=1}^{4} \Pr(\text{k-th flip is Heads}) \]

\[ = \sum_{k=1}^{4} \frac{1}{2} = 2 \]