CS200 - Worksheet 5

1. Let $T(n)$ be the number of strings in $\{0, 1, 2\}^n$ that do *not* contain two consecutive zeros. Write a recurrence relation for $T(n)$

   Solution Base case: $T(1) = 3$, $T(2) = 8$.

   Recurrence relation: There are 3 cases for the final position of the string: $0, 1, 2$. In the case that the final position is 1 or 2, then we must have that the first $n-1$ positions don't have two consecutive zeros, which is $T(n-1)$. This give $2T(n-1)$ options.

   Now if the final position is 0, we must have that there are not two consecutive zeros in the first $n-1$ positions, AND, the $(n-1)$st position must not be zero. This means that the $(n-1)$st position can be either 1 or 2, and the preceding $n-2$ positions can not have two consecutive zeros. This gives $2T(n-2)$ options.

   Thus $T(n) = 2T(n-1) + 2T(n-2)$

2. **[Challenge]** Let $T(n)$ be the number of strings in $\{0, 1, 2\}^n$ that do not have 2 consecutive 0's or 2 consecutive 1's. Create a recurrence relation for $T(n)$.

   Solution Let $T_1(n)$ be the number of strings of length n that do not contain 2 consecutive 0s or 2 consecutive 1's and that ends in 1. We define $T_0(n)$ and $T_2(n)$ similarly. Then there are three options: the final position can be 0, 1, or 2.

   If the final position is 2, then there are $T(n-1)$ options for the first $n-1$ positions.

   If the final position is 1, then the number of choices for the first $n-1$ positions is $T_0(n-1) + T_2(n-1)$ because we can't have a 1 in the final position. If the final position is 0, then the number of choices for the first $n-1$ positions is $T_1(n-1) + T_2(n-1)$ because we can't have a 0 in the final position.

   Adding everything up, we have

   $$T(n) = T(n-1) + T_0(n-1) + T_2(n-1) + T_1(n-1) + T_2(n-1) = 2T(n-1) + T_2(n-1). \tag{1}$$

   The number of ways to have no consecutive 0's or 1's in the first $n-1$ positions and also end in a 2 is just the number of ways to have no consecutive 0's or 1's in the first n-2 positions. Thus

   $$T(n) = 2T(n-1) + T(n-2). \tag{2}$$

   Base case: $T(1) = 3$, $T(2) = 7$.

3. Suppose you have a coin that has a changing probability of getting heads. When you toss it the ith time, the probability of getting heads is $1/2^i$. If you flip the coin an infinite number of times, how many heads would you expect to see?

Solution Let X be the random variable that is the number of heads. Let X_i be an indicator random variable that takes value 1 if the ith toss, the coin comes up heads. Then

$$X = \sum_{i=1}^{\infty} X_i, \tag{3}$$

so using linearity of expectation.

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{E}[X_i] = \sum_{i=1}^{\infty} \mathbb{E}Pr(i\text{th outcome is heads}) = \sum_{i=1}^{\infty} \frac{1}{2^i}. \tag{4}$$

Using the formula for a geometric series, this is equal to

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \frac{1}{2^i} = \sum_{i=0}^{\infty} \frac{1}{2^i} - 1 = \frac{1}{1-1/2} - 1 = 1. \tag{5}$$

4. **[Challenge]** Let $[n] = \{1, 2, 3, \ldots, n\}$. Given a permutation of the elements of $[n]$, an inversion is an ordered pair (i, j) with $i, j \in [n]$, such that $i < j$, but j precedes i in the permutation. For instance consider the set [5], and the permutation $(3, 5, 1, 4, 2)$. There are six inversions in this permutation:

$$(1,3), (1,5), (2,3), (2,4), (2,5), (4,5). \tag{6}$$

If a permutation is uniformly at random from among all permutations, what is the expected number of inversions? (Hint - use indicator random variables! To figure out the probability of the indicator event happening, try a small example, like [3].)

Solution Let $X_{i,j}$ be the indicator random variable that takes value 1 if i and j are inverted. Let X be the random variable that is the total number of inversions. Then

$$X = \sum_{i,j:i<j} X_{i,j} \tag{7}$$

Using Linearity of Expectation,

$$\mathbb{E}[X] = \sum_{i,j:i<j} \mathbb{E}[X_{i,j}] = \sum_{i,j:i<j} Pr(i, j \text{ inverted}). \tag{8}$$

Now i is just as likely to be before j as after j. To see this, just note that for every permutation where i is in front of j, there is another permutation where all of the other elements are in the same positions, but j is in front of i. Thus the probability that i and j are inverted is $1/2$. So

$$\mathbb{E}[X] = \sum_{i,j:i<j} \mathbb{E}[X_{i,j}] = \sum_{i,j:i<j} \frac{1}{2} = \binom{n}{2} \times \frac{1}{2} = \frac{n(n-1)}{4}. \tag{9}$$