1. Let $T(n)$ be the number of strings in $\{0, 1, 2\}^n$ that do not contain two consecutive zeros. Write a recurrence relation for $T(n)$

**Solution** Base case: $T(1) = 0$, $T(2) = 8$.

Recurrence relation: There are 3 cases for the final position of the string: 0, 1, 2. In the case that the final position is 1 or 2, then we must have that the first $n - 1$ positions don’t have two consecutive zeros, which is $T(n - 1)$. This give $2T(n - 1)$ options.

Now if the final position is 0, we must have that there are not two consecutive zeros in the first $n - 1$ positions, AND, the $(n - 1)$st position must not be zero. This means that the $(n - 1)$st position can be either 1 or 2, and the preceding $n - 2$ positions can not have two consecutive zeros. This gives $2T(n - 2)$ options.

Thus $T(n) = 2T(n - 1) + 2T(n - 2)$.

2. **[Challenge]** Let $T(n)$ be the number of strings in $\{0, 1, 2\}^n$ that do not have 2 consecutive 0’s or 2 consecutive 1’s. Create a recurrence relation for $T(n)$.

**Solution** Let $T_1(n)$ be the number of strings of length $n$ that do not contain 2 consecutive 0s or 2 consecutive 1’s and that ends in 1. We define $T_0(n)$ and $T_2(n)$ similarly. Then there are three options: the final position can be 0, 1, or 2.

If the final position is 2, then there are $T(n - 1)$ options for the first $n - 1$ positions.

If the final position is 1, then the number of choices for the first $n - 1$ positions is $T_0(n - 1) + T_2(n - 1)$ because we can’t have a 1 in the final position. If the final position is 0, then the number of choices for the first $n - 1$ positions is $T_1(n - 1) + T_2(n - 1)$ because we can’t have a 0 in the final position.

Adding everything up, we have

$$T(n) = T(n - 1) + T_0(n - 1) + T_2(n - 1) + T_1(n - 1) + T_2(n - 1) = 2T(n - 1) + T_2(n - 1).$$

The number of ways to have no consecutive 0’s or 1’s in the first $n - 1$ positions and also end in a 2 is just the number of ways to have no consecutive 0’s or 1’s in the first $n-2$ positions. Thus

$$T(n) = 2T(n - 1) + T(n - 2).$$

Base case: $T(1) = 3$, $T(2) = 7$. 


3. Suppose you have a coin that has a changing probability of getting heads. When you toss it the $i^{th}$ time, the probability of getting heads is $1/2^i$. If you flip the coin an infinite number of times, how many heads would you expect to see?

**Solution** Let $X$ be the random variable that is the number of heads. Let $X_i$ be an indicator random variable that takes value 1 if the $i^{th}$ toss, the coin comes up heads. Then

$$X = \sum_{i=1}^{\infty} X_i,$$

so using linearity of expectation.

$$E[X] = \sum_{i=1}^{\infty} E[X_i] = \sum_{i=1}^{\infty} EPr(i^{th} outcome is heads) = \sum_{i=1}^{\infty} \frac{1}{2^i}. \quad (4)$$

Using the formula for a geometric series, this is equal to

$$E[X] = \sum_{i=1}^{\infty} \frac{1}{2^i} = \sum_{i=0}^{\infty} \frac{1}{2^i} - 1 = \frac{1}{1-1/2} - 1 = 1. \quad (5)$$

4. [Challenge] Let $[n] = \{1, 2, 3, \ldots, n\}$. Given a permutation of the elements of $[n]$, an inversion is an ordered pair $(i, j)$ with $i, j \in [n]$, such that $i < j$, but $j$ precedes $i$ in the permutation. For instance consider the set $[5]$, and the permutation $(3, 5, 1, 4, 2)$. There are six inversions in this permutation:

$$(1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (4, 5). \quad (6)$$

If a permutation is uniformly at random from among all permutations, what is the expected number of inversions? (Hint - use indicator random variables! To figure out the probability of the indicator event happening, try a small example, like $[3]$.)

**Solution** Let $X_{i,j}$ be the indicator random variable that takes value 1 if $i$ and $j$ are inverted. Let $X$ be the random variable that is the total number of inversions. Then

$$X = \sum_{i,j:i<j} X_{i,j} \quad (7)$$

Using Linearity of Expectation,

$$E[X] = \sum_{i,j:i<j} E[X_{i,j}] = \sum_{i,j:i<j} EPr(i, j \text{ inverted}). \quad (8)$$

Now $i$ is just as likely to be before $j$ as after $j$. To see this, just note that for every permutation where $i$ is in front of $j$, there is another permutation where all of the other elements are in the same positions, but $j$ is in front of $i$. Thus the probability that $i$ and $j$ are inverted is $1/2$. So

$$E[X] = \sum_{i,j:i<j} E[X_{i,j}] = \sum_{i,j:i<j} \frac{1}{2} = \left( \begin{array}{c} n \\ 2 \end{array} \right) \times \frac{1}{2} = \frac{n(n-1)}{4}. \quad (9)$$