Announcements

- I am away Wed./Fri
  - Friday: in-class quiz (Prof. Andrews)
  - I will post video lectures to website. You may want to come to class and watch discuss together
- Self-Grade/Reflections: Due by 3 pm Wed to my mailbox or submitted on Canvas
  ^ outside MBH 633
- Extra Office Hours on Wed, Thurs, Friday
  Book at Calendly.com/skimmel
  Will use ZOOM. I will send Meeting ID
  Go to Middlebury.zoom.us
  Click Join & put in ID
- Extra office hours MT drop-in

∀x ∃y
↑
Nemesis gets to choose y from any in domain
→ you get to choose one specific y based on x from nemesis and prove true

P(x,y) = x+y=10
Can get rid of y in predicate using quantifier:
P(x) = ∀y, x+y=10
Direct Proof

- Usually used to prove implication like $P \rightarrow Q$.

Structure:
Assume $P$. Explain, explain, ... explain. Therefore $Q$

"By definition..."

Also used to prove universal implication: $\forall x (P(x) \rightarrow Q(x))$

Structure:
Let $x$ be an arbitrary element of the domain
Assume $P(x)$
Explain, explain, explain.
Therefore $Q(x)$
Q: Use a direct proof to show: For all $a, b, c \in \mathbb{Z}$, if $a | b$ and $b | c$ then $a | c$.

($a | b$ means $a$ divides $b$, that $\exists e \in \mathbb{Z}: ae = b$.

Let $a, b, c \in \mathbb{Z}$. Assume $a | b$ and $b | c$. This means $\exists e, f \in \mathbb{Z}$ such that $b = ae$ and $c = bf$. Then

$$c = bf = (ae)f = a(ef).$$

But this means $a | c$, since $c = a \cdot k$, for $k = ef$, an integer.
\[ P \iff Q \text{ if and only if} \]

\[(P \rightarrow Q) \land (Q \rightarrow P) \text{ logically equivalent to} P \iff Q\]

Structure

For the forward direction, \([\text{Proof of } P \rightarrow Q]\)

For the backward direction \([\text{Proof of } Q \rightarrow P]\)
Direct Proof - Proof by Contrapositive

Usually used to prove implication like $P \implies Q$.

Recall: $P \implies Q$ is logically equivalent to $\neg Q \implies \neg P$

Structure:

We prove the contrapositive.

Assume $\neg Q$. Explain, explain, ... explain. Therefore $\neg P$

Also used to prove universal implication: $\forall x (P(x) \implies Q(x))$

Structure:

Let $x$ be [an arbitrary] element of the domain

We prove the contrapositive.

Assume $\neg Q(x)$

Explain, explain, explain.

Therefore $\neg P(x)$

(See example on hw!)
What if statement is not of the form \( P \rightarrow Q \)?

What if just have \( P \)?

Suppose you can show

\[
\begin{array}{c|c|c|c|c|c}
P & Q & \Gamma P & \Gamma Q & \Gamma P \rightarrow Q & \Gamma P \rightarrow \neg Q \\
\hline
T & T & F & F & T & T \\
T & F & F & T & T & T \\
F & T & T & F & T & F \\
F & F & T & T & T & T \\
\end{array}
\]

\[\neg \Gamma P \rightarrow \neg Q \]

"Proof needs to do two things" (direct)

\[\Gamma P \rightarrow Q\]

\[\Gamma P \rightarrow \neg Q\]

\[\therefore P\]

"Structure: (Prove \( P \))"

"For contradiction, assume \( \neg P \)"

"or"

"We proceed by contradiction. We assume \( \neg P \)."

\[
\begin{array}{c|c|c|c|c|c}
P & Q & \Gamma P & \Gamma Q & \Gamma P \rightarrow Q & \Gamma P \rightarrow \neg Q \\
\hline
T & T & F & F & T & T \\
T & F & F & T & T & T \\
F & T & T & F & T & F \\
F & F & T & T & T & T \\
\end{array}
\]

Therefore, \( Q \)

However

\[\therefore \neg Q, a \text{ contradiction. Thus, } P.\]