1. Let $G$ be the following graph, and let $A_G$ be an adjacency list representation of the $G$ with the adjacency list for each vertex in alphabetical order.

Consider the following slight variation to breadth-first-search:

**Algorithm 1: BFSish($A, s$)**

- **Input:** Adjacency list $A$ for a graph $(V, E)$ and vertex $s$
- **Output:** An integer array $L$ of length $|V|$.

// Initialize array of explored vertices and array $L$
1. $X[v] = 0 \forall v \in V$;
2. $L[v] = \infty \forall v \in V$;
3. $X[s] = 1$;
4. $L[s] = 0$;

// Initialize Queue $A$
5. $A = \{}$;
6. $A.add(s)$;
7. while $A$ is not empty do
8. \hspace{1em} $v = A.pop$;
9. \hspace{1em} for each edge $\{v, w\}$ do
10. \hspace{2em} if $X[w] == 0$ then
11. \hspace{3em} $X[w] = 1$;
12. \hspace{3em} $A.add(w)$;
13. \hspace{3em} $L[w] = L[v] + 1$;
14. \hspace{2em} end
15. \hspace{1em} end
16. end
17. return $L$
(a) [6 points] In what order does $\text{BFSish}(A_G, a)$ explore the nodes of the graph $G$? (Remember lists of the adjacency list representation are in alphabetical order, so the for loop in line 9 will look at vertices in alphabetical order.)

(b) [6 points] What are the values in the array $L$ that is returned when $\text{BFSish}(A_G, a)$ is implemented?

(c) [3 points] Considering your answer to part b, what does it seem like the algorithm is doing? What is the meaning of $L$? Think about the relationship between $a$, $b$, and $L[b]$.

2. Geometric Series

(a) [11 points] Use induction to prove that for all $n \in \mathbb{N}$, and any $r \in \mathbb{R}$ such that $r \neq 0$

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$  \hspace{1cm} (1)

(b) [2 points] What does the sum evaluate to when $r = 1$?

3. Recurrence Relations

(a) Let $G(n)$ be the number of bit strings of length $n$ that have two consecutive zeros. (A bit string of length $n$ is an element of $\{0, 1\}^n$.) Consider a recurrence relation for $G(n)$.

i. [3 points] What are the initial conditions for the recurrence relation?

ii. [6 points] What are the recursive conditions for the recurrence relation?

iii. [6 points] Use the recurrence relation to calculate $T(5)$.

(b) Consider the following variant to the Tower of Hanoi. We start with all disks on peg 1 and want to move them to peg 3, but we cannot move a disk directly between peg 1 and peg 3. Instead, we can only move disks from peg 1 to peg 2, or from peg 2 to peg 3. So in the case of $n = 1$ disk, we have to first move the disk from 1 to 2, and then from 2 to 3. Let $T(n)$ be the number of moves required to shift a stack of $n$ disks from peg 1 to peg 3, if as usual, we can not put a larger disk on top of a smaller disk. Consider creating a recurrence relation for $T(n)$.

i. [3 points] What are the initial conditions for the recurrence relation?

ii. [6 points] What are the recursive conditions for the recurrence relation? (Explain)

iii. [6 points] Use the iterative method to solve for $T(n)$ and give a big-O bound on $T(n)$.

4. Consider rolling 5 dice. Let $X_{i,j}$ be an indicator random variable that takes value 1 if the $i$th di has outcome $j$ (and takes value 0 otherwise).

(a) [3 points] What is the sample space? What is its size?

(b) [3 points] Let $X$ be the random variable that is the sum of all of the values shown on the dice. If the outcome of the rolls is $s = (4, 2, 4, 5, 1)$, what is $X(s)$? What is $X_{3,4}(s)$? What is $X_{4,3}(s)$?

(c) [3 points] Write $X$ in terms of a weighted sum of the variables $X_{i,j}$. 

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(d) [3 points] What is $E[X_{i,j}]$?

(e) [3 points] Use linearity of expectation to determine the average value of the sum of all values shown on the dice.

5. Suppose a group of $n$ people each order a different flavor of ice cream at an ice cream shop. Suppose the server didn’t keep track of who ordered which flavor, and just handed the ice cream out randomly.

(a) 3 points Let $X$ a random variable that is the number of people who got handed the correct flavor. Let $X_i$ be the indicator random variable that takes value 1 if person $i$ gets the correct flavor (and 0 otherwise.) Write $X$ in terms of a sum of the $X_i$.

(b) 6 points Use linearity of expectation to determine the average number of people who get the correct flavor.

6. How long did you spend on this homework?