Intro to Algorithm Complexity

Important function: worst case time complexity of an algorithm in C.S.

\[ T: \mathbb{N} \rightarrow \mathbb{N} \]

- input size
- \# of operations performed by algorithm in worst case.

unless parallel computing, this tells you the time the computer will take to run the algorithm. Just multiply by time to do 1 operation.

Linear Search

- Input: \((a_1, a_2, \ldots, a_n)\), \(x\) \[ \Delta \] Input size is \(n\)
- Output: \(j\) if \(a_j = x\), 0 otherwise

1) \(i = 1\)
2) while \((i \leq n\) and \(x \neq a_i\))
3) \(i = i + 1\)
4) if \(i \leq n\):
   return \(i\)
5) else:
   return 0

Q: What is \(T(n)\) for linear search? (Hint: \(n\) is not correct)

Report by
By group.
Issues:
- too fine-grained / detailed
  - different computers might do operations differently
  - when \( n \) gets large, don't care about \( 100000 \) vs \( 100001 \)
- too difficult to count every operation

Big-O to Rescue!

\[ f(x) \leq C |g(x)| \]

**Def:** Let \( f, g : \mathbb{Z} \rightarrow \mathbb{R} \). Then \( f(x) \) is \( O(g(x)) \) if there exist constants \( k \in \mathbb{Z} \) and \( C \in \mathbb{R} \) such that when \( x \geq k \), then

**Ex:** \( 2n+4 \) is \( O(n) \)

Let's choose \( C = 3 \).
When cross? \( 3n = 2n + 4 \)
\[ n = 4 \]

Proof:
When \( n \geq 4 \), we have
\[ 2n + 4 \leq 2n + n = 3n, \] so
\[ 2n + 4 = O(n) \text{ with } k = 4, C = 3 \]
Q: Prove other functions for linear search # of operations is $O(n)$

Starting to see why big-O is good for algorithm time complexity:
- Small differences in how you calculate don't matter
- Not too fine grained

However big-O is only upper bound:

ex: $7x + 1$ is $O(x^2)$

Pf: $7x + 1 \leq 7x + x$ for all $x \geq 1$
$7x + x = 8x \leq x^2$ for all $x \geq 8$
Thus with $k = 8$, $C = 1$, $7x + 1 = O(x^2)$
Q: Prove $10x^2$ is not $O(x)$. This means there do not exist constants $K, C$, such that $10x^2 < Cx$ for all $x \geq K$.

Pf: For contradiction, assume $K, C$ exist. Then for all $x \geq K$, we have 
$$10x^2 < Cx$$

When $x > 0$, we have $x \leq \frac{C}{10}$. Thus, this inequality holds only when $0 < x \leq \frac{C}{10}$, which contradicts that it should hold for all $x \geq K$. 
