Relationships between spanning sets, linearly independent sets, and bases

- A set \( S \) is linearly dependent in \( V \) if and only if one of its vectors is a linear combination of the rest.

- If \( S \) is linearly independent in \( V \) and \( W \) spans \( V \), then \( |S| \leq |W| \).
  - Method of proof: incrementally replace vectors of \( S \) into \( W \) (maintaining it as a spanning set) via careful use of hypotheses and replacement rules.
  - Consequences:
    - If \( S \) spans \( V \), \( \mathcal{B}_i \) is linearly independent in \( V \), and \( \mathcal{B}_k \) is a basis for \( V \), then \( |\mathcal{B}_i| \leq |\mathcal{B}_k| \leq |\mathcal{B}_n| \).
    - All bases for \( V \) have the same size (because one is l.i. and the other spans \( V \), and vice-versa).

Basis and dimension

- \( \dim V \) is defined as the size of any basis for \( V \).
  - This is intrinsic to \( V \), because all bases for \( V \) have the same size.

- Any linearly independent collection in \( V \) can be extended to a basis for \( V \).
  - Method of proof: iteratively insert a vector not in the collection’s span, until the collection spans \( V \).
  - Consequences:
    - A collection of size larger than \( \dim V \) can’t be linearly independent in \( V \).
    - A linearly independent collection having size \( \dim V \) must be a basis for \( V \).
    - Every finite-dimensional vector space has a basis (extend the [l.i.] empty collection to a basis).

- Any spanning set for \( V \) can be reduced to a basis for \( V \).
  - Method of proof: iteratively remove a vector that’s a linear combination of the rest, until the collection is linearly independent in \( V \).
  - Consequences:
    - A collection of size smaller than \( \dim V \) can’t span \( V \).
    - A spanning set of \( V \) having size \( \dim V \) must be a basis for \( V \).

Bases and coordinate mappings

Suppose that \( \mathcal{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) \) is an ordered basis for a vector space \( V \).

We then have a corresponding linear combination function \( [\mathcal{B}] : \mathbb{R}^n \to V \) and \( \mathcal{B} \)-coordinate function \( [\mathcal{B}]^{-1} : V \to \mathbb{R}^n \).

- \([\mathcal{B}] : \mathbb{R}^n \to V\) maps each coordinate [coefficient] vector in \( \mathbb{R}^n \) to the corresponding linear combination of \( \mathcal{B} \) in \( V \).
  - Because \( [\mathcal{B}] \) is a basis for \( V \), \( [\mathcal{B}] \) pairs each column vector in \( \mathbb{R}^n \) with one vector of \( V \) and vice-versa.
  - \([\mathcal{B}] \) is bijective and linear, so it gives a vector space isomorphism from \( \mathbb{R}^n \) to \( V \).

- \([\mathcal{B}]^{-1} : V \to \mathbb{R}^n \) is the inverse of the linear combination function \([\mathcal{B}] \).
  - \([\mathcal{B}]^{-1} \) maps each vector in \( V \) to its \( \mathcal{B} \)-coordinates (the vector of coefficients that build \( \vec{v} \) as a l.c. of \( \mathcal{B} \)).
  - \([\mathcal{B}]^{-1} \) is also bijective and linear, and thus also gives an isomorphism (from \( V \) to \( \mathbb{R}^n \)).

- Being inverse functions, \([\mathcal{B}][\mathcal{B}]^{-1} : V \to V \) and \([\mathcal{B}]^{-1}[\mathcal{B}] : \mathbb{R}^n \to \mathbb{R}^n \) are just the identity maps (i.e., these functions cancel each other).

- When dealing with bases, coordinates, and the functions \([\mathcal{B}] \) and \([\mathcal{B}]^{-1} \), always be alert to which vector space and basis you’re dealing with and what the entries of a given column vector represent; with this kept in mind, all that these functions do is either form a linear combination of \( \mathcal{B} \) or find the coefficients to build a vector as a linear combination of \( \mathcal{B} \).

Common bases

- The standard basis for \( \mathbb{R}^m \) is \( (\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m) \), where \( \vec{e}_j \) has all zero entries except for a 1 in the \( j \)-th position.

- For the subspace \( P_n(x) = \{ a_0 + a_1 x + \cdots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{R} \} \) of \( \mathbb{R}[x] \), we have the basis \( (1, x, x^2, \ldots, x^n) \).
Vector space isomorphism and the FTVS

- An **isomorphism** from $V$ to $W$ is a bijection $\varphi : V \rightarrow W$ with the property of **linearity**:

$$\forall \vec{v}_1, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \ldots, \alpha_n, \quad \varphi(\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n) = \alpha_1\varphi(\vec{v}_1) + \cdots + \alpha_n\varphi(\vec{v}_n)$$

- The **inverse** of an isomorphism is an isomorphism, and the **composition** of two isomorphisms is an isomorphism.

- Two vector spaces $V$ and $W$ are **isomorphic** if there exists an isomorphism from $V$ to $W$.

- For $V$ and $W$ to be isomorphic means that they, in a very literal sense, have the “same shape” as each other—they are mathematically equivalent as vector spaces.

- An isomorphism (and its inverse) allow us to translate any linear algebra problem in $V$ to one in $W$ and vice-versa:
  - Each vector in $V$ has a corresponding vector in $W$, and vice-versa.
  - Each linear combination in $V$ has a corresponding linear combination in $W$, and vice-versa.
  - Each spanning set, linearly independent set, or basis in $V$ has a corresponding spanning set, linearly independent set, or basis in $W$, and vice-versa.
  - Any other problem formed from the concept of linear combination can be translated from $V$ to $W$, and vice-versa.

- The **FTVS**: Any $n$-dimensional vector space $V$ [over $\mathbb{R}$] is isomorphic to $\mathbb{R}^n$.
  - Method of proof: Taking any basis $B = (\vec{v}_1, \ldots, \vec{v}_n)$ for $V$ gives us a coordinate isomorphism $[B]^{-1} : V \rightarrow \mathbb{R}^n$.

- Corollary to the FTVS: Any two vector spaces $V$ and $W$ of the same dimensions [over the same field] are isomorphic.
  - Method of proof: Take bases for $V$ and $W$, giving coordinate isomorphisms for each; invert one and compose to obtain an isomorphism from $V$ to $W$.
  - Consequence: dimension is the fundamental property intrinsic to an abstract vector space (considered up to isomorphism).

- Computation via the FTVS: We can translate any linear algebra problem in any finite-dimensional vector space into a column vector problem, simply by choosing a basis for the vector space and working with coordinates (then translating our result).

**Column-vector methods**

Suppose that we have collections of **column vectors** $C, D$ in $\mathbb{R}^m$.

**Spanning**

- $\vec{v} \in \text{span } C$ $\iff$ $[C | \vec{v}]$ is consistent.
- $\vec{v} \notin \text{span } C$ $\iff$ $[C | \vec{v}]$ is inconsistent.
- span $C \supset$ span $D$ $\iff$ $[C | D]$ is consistent.
- span $C =$ span $D$ $\iff$ $[C | D]$ and $[D | C]$ and are both consistent.
- $C$ spans $\mathbb{R}^m$ $\iff$ $[C]$ gives a pivot in every row.

**Linear Independence**

- $C$ is linearly independent $\iff$ $[C]$ gives a pivot in every column. $C$ is linearly dependent if not.
- Linear relations on $C$ are just solutions of the system $[C]$. (to find a nontrivial linear relation, set some free variable to 1)

**Bases**

- $C$ is a basis for $\mathbb{R}^m$ $\iff$ $[C]$ gives a pivot in every row and column.
- To find a basis for span $C$, take the vectors of $C$ that give pivots in $[C]$.
- To extend a l.i. collection $C$ to a basis for $\mathbb{R}^m$, append the standard basis to $C$ and keep the ones that give pivots.

**Coordinates**

Suppose that $B$ is a basis for $\mathbb{R}^m$.

- Compute $[B]x$ as usual, simply by using the entries of $x$ to form a linear combination of $B$.
- Compute $[B]^{-1}x$ by solving $[B | x]$ and expressing the solution as a column vector.