A reduction from set $A$ to set $B$ is a computable function $\sigma : A \rightarrow B$ such that

$$x \in A \iff \sigma(x) \in B.$$ 

If $A$ reduces to $B$ via $\sigma$ we write $A \leq_{\sigma} B$.

We use reductions to prove certain sets are not recursive or not r.e. In class we showed:

1. If $A \leq B$ and $A$ is not recursive, then $B$ is not recursive.
2. If $A \leq B$ and $A$ is not r.e., then $B$ is not r.e.

So if we have a set $A$ that we know to be not recursive (e.g., $HP = \{M\#x \mid M \text{ halts on } x\}$) and we can reduce that set $A$ to another set $B$, then we have shown $B$ is not recursive. Similarly a reduction from a non-r.e. set $A$ (e.g., $\overline{HP} = \{M\#x \mid M \text{ loops on } x\}$) to another set $B$ would show that $B$ is not r.e.

Use the technique of reduction for each of the following:

1. Let $ALL = \{M \mid L(M) \text{ accepts all strings}\}$. Show $ALL$ is not recursive.

2. Let $REC = \{M \mid L(M) \text{ is recursive}\}$. Show $REC$ is not r.e.

3. Let $INF = \{M \mid L(M) \text{ is infinite}\}$. Show $INF$ is not r.e.

4. Let $REC = \{M \mid L(M) \text{ is recursive}\}$. Show $REC$ is not recursive.

5. Let $A$ and $B$ be sets of Turing machines such that

$$A = \{M \mid L(M) = \Sigma^*\} \text{ and } B = \{N \mid L(N) = \{0^n1^n \mid n \geq 1\}\}.$$ 

Show that $A \leq B$. 