The Kleene Hierarchy

\[ \exists \forall \quad \sum_2 \quad \Pi_2 \quad \Delta_2 \quad \sum_1 \quad \Pi_1 \quad \Delta_3 \quad \sum_3 \quad \Pi_3 \]

- \exists (r.e.)
- \forall (co-r.e.)
- \forall
- \exists

\begin{align*}
\text{r.e.} & : \text{finite}, \text{all, NUL} \\
\text{co-r.e.} & : \text{recursive, MP, NUL} \\
\text{recursive} & : \text{co-rec, CFL, REG, COF}
\end{align*}
\[\text{HP} = \{M\#x \mid M \text{ halts on } x\} = \{M\#x \mid \exists t: M \text{ halts on } x \text{ in } t \text{ steps}\}\]

\[\text{MP} = \{M\#x \mid M \text{ accepts } x\} = \{M\#x \mid \exists t: M \text{ accepts } x \text{ in } t \text{ steps}\}\]

\[\text{NUL} = \{M \mid L(M) = \emptyset\} = \{M \mid \forall x, t: M \text{ has not accepted } x \text{ in } t \text{ steps}\}\]

\[\text{FIN} = \{M \mid L(M) \text{ is finite}\}\]

\[= \{M \mid \exists n \forall x, t: |x| > n: M \text{ has not accepted } x \text{ in } t \text{ steps}\}\]

\[\text{INF} = \{M \mid L(M) \text{ is infinite}\}\]

\[= \{M \mid \forall n \exists y, t: |y| > n: M \text{ accepts } y \text{ in } t \text{ steps}\}\]

\[\text{ALL} = \{M \mid L(M) = \Sigma^*\} = \{M \mid \forall x \exists t: M \text{ accepts } x \text{ in } t \text{ steps}\}\]

\[\text{REC} = \{M \mid L(M) \text{ is recursive}\}\]

\[= \{M \mid \exists \text{ total TM } N \forall x \exists t: x \in L(N) \iff M \text{ accepts } x \text{ in } t \text{ steps}\}\]

\[\text{CFL} = \{M \mid L(M) \text{ is context free}\}\]

\[= \{M \mid \exists \text{ PDA } N \forall x \exists t: x \in L(N) \iff M \text{ accepts } x \text{ in } t \text{ steps}\}\]

\[\text{REG} = \{M \mid L(M) \text{ is regular}\}\]

\[= \{M \mid \exists \text{ DFA } D \forall x \exists t: x \in L(D) \iff M \text{ accepts } x \text{ in } t \text{ steps}\}\]

\[\text{COF} = \{M \mid \neg L(M) \text{ is finite}\}\]

\[= \{M \mid \exists n \forall x, |x| > n, \exists t: M \text{ accepts } x \text{ in } t \text{ steps}\}\]