General Instructions

1. Please be concise. Type your answers or write them clearly.

2. If you worked with or discussed with someone else, including the TAs or the professor, please list their names for each problem as appropriate.

3. Turn in your work at the beginning of class on the due date.

Reading

Book of Proof Chapters 1 and 2.

Programming Problem

Most weeks the homework will include a programming problem, which you can complete using a programming language of your choice (e.g., Python, C, C++, Java, among others). While there is no programming problem per se this week, we ask you to consider how you would work with sets in a programming context.

1. Name a programming language you would use if you had a programming task that involved operations on simple sets.

2. Peruse some resources that describe the language you chose. Does the language provide a Set datatype (built-in or through a library)? If so, name a few operations that are provided or that you would want to define, and give at least three sample usages. For example, in the language Scheme, an example would be \(\text{union '(a b c) '(b c d e)} \Rightarrow (a b c d e)\).

Written Problems

1. For each function below from \(\mathbb{Z}\) to \(\mathbb{Z}\), state whether it is 1-to-1, onto, or a bijection. Note that a function is a bijection iff it is 1-to-1 and onto. (For instance, an answer might look like “\(f(n)\) is not 1-to-1, it is onto, and it is not a bijection”.) No justification is necessary.

   (a) \(f(n) = n - 1\)
   (b) \(f(n) = n^2 + 1\)
   (c) \(f(n) = n^3\)
   (d) \(f(n) = \lceil n/2 \rceil\)

2. The following sets are described in set builder notation. Describe each of them in roster notation, instead. The following symbols are used: \(\mathbb{Z}\) denotes the set of integers; \(\mathbb{R}\) denotes the set of real numbers; \(\mathbb{N}\) denotes the set of natural numbers, i.e., \(\mathbb{N} = \{0, 1, 2, \ldots\}\).

   (a) \(\{z \mid z \in \mathbb{Z} \text{ and } 3 < z < 6\}\).
   (b) \(\{z \mid z \in \{a, c, e\} \text{ and } z \neq c\}\).
   (c) \(\{A \mid A \subseteq \{a, c, e\} \text{ and } |A| \neq 2\}\)
   (d) \(\{r \mid r \in \mathbb{R} \text{ and } r = r^2\}\)
3. Let $U$, the universal set, be the set of integers from 2 to 12 inclusive, and let $A = \{4, 6, 7, 9\}$, $B = \{2, 3, 4, 5, 6\}$, and $C = \{4, 6, 8, 12\}$. For each of the following sets, list its elements in roster notation.

(a) $A \cup B$
(b) $C \cap B$
(c) $(A \cup C) \cap B$
(d) $A \cap B$
(e) $(C \setminus A) \cap B$
(f) Draw the Venn diagram for the three sets $A, B, C$ within $U$.

4. More practice with set operations and the set builder notation.

(a) In set builder notation express the set of all pairs of integers such that the numbers in a pair are non-zero, have opposite signs, and the magnitude of one of them is the square of the magnitude of the other.

(b) For any two sets $A$ and $B$, define $op(A, B)$ as the set $\{c \mid \forall a \in A \exists b \in B, \ c = a + b\}$. In all parts below, you are given $A$ and $B$ and are asked for $op(A, B)$. If $op(A, B)$ is a finite set, directly state what it is; else state it in the simplest form possible. You don’t need to provide any justification.

i. If $A = \{2, 3, 5\}$ and $B = \{1, 2, 3, 8, 9, 10, 11, 12, 13, 14, 50, 51, 52, 53\}$, what is $op(A, B)$?

ii. If $A = \{2, 3, 5\}$ and $B$ is the set of even integers, what is $op(A, B)$?

iii. If $A = \{2, 3, 5\}$ and $B = \mathbb{Z}^+$, the set of positive integers, what is $op(A, B)$?

iv. If $A = \{2, 3, 5\}$ and $B = \mathbb{Z}$, what is $op(A, B)$?

5. Simplify each of the following expressions, where $A$ and $B$ are arbitrary sets, \emptyset is the empty set, and $U$ is the universal set. Hint: each answer to (a)-(h) is one of $A, U,$ or $\emptyset$. Just write down the answer: no proof needed, no steps need be shown.

(a) $A \cap U$
(b) $A \cap \emptyset$
(c) $A \cup U$
(d) $A \cup \emptyset$
(e) $A \cup A$
(f) $A \cap A$
(g) $A \cup \overline{A}$
(h) $A \cap \overline{A}$
(i) $A \cap (A \cup B)$
6. Verify the following set equalities using Venn diagrams.

(a) \( A \cap B = A \cup B \)
(b) \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \)
(c) \( (A - B) - C = (A - C) - (B - C) \)

7. Simplify each of the following statements, where \( p \) denotes a proposition, and \( T \) and \( F \) are the Boolean constants \textit{true} and \textit{false}. Hint: each answer is one of \( p, T, \) or \( F \).

(a) \( T \land p \)
(b) \( F \land p \)
(c) \( T \lor p \)
(d) \( F \lor p \)
(e) \( p \lor p \)
(f) \( p \land p \)
(g) \( p \lor \neg p \)
(h) \( p \land \neg p \)

8. The following are some important equivalences:

(a) \( p \rightarrow q \equiv \neg q \rightarrow \neg p \) \hspace{1cm} \text{(a statement is equivalent to its contrapositive)}
(b) \( \neg (p \land q) \equiv \neg p \lor \neg q \) \hspace{1cm} \text{(DeMorgan’s Law)}
(c) \( \neg (p \lor q) \equiv \neg p \land \neg q \) \hspace{1cm} \text{(DeMorgan’s Law)}
(d) \( p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \) \hspace{1cm} \text{\((\land \text{ distributes over } \lor)\)}
(e) \( p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \) \hspace{1cm} \text{\((\lor \text{ distributes over } \land)\)}

Give an informal justification for each of (a), (b), and (c). Show (d) through truth table. When writing the truth table, be sure to include columns for subexpressions, such as \( p \land q \). The following is an example truth table for (e) showing in the last column the equivalence of output columns 2 and 5.

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<th>( p \lor (q \land r) )</th>
<th>( p \lor q )</th>
<th>( p \lor r )</th>
<th>( (p \lor q) \land (p \lor r) )</th>
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9. Approximately how many hours did you spend on this assignment, including reading?