1 Summation Notation

When we wish to make a sum of many number, the following notation is used:

\[ \sum_{i=1}^{n} f(i) := f(1) + f(2) + f(3) + \ldots + f(n-1) + f(n). \]

In summation notation, as this is called, the variable \( i \) is an integer and the function \( f \) is evaluated at all integers between the lower and upper summation limits.

Examples:
1. \( \sum_{3}^{5} i^2 = 3^2 + 4^2 + 5^2 = 50 \)
2. \( \sum_{1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55 \)
3. \( \sum_{1}^{3} 1 + 1 + 1 = 3 \)
4. \( \sum_{0}^{2} \sin(i \frac{\pi}{2}) = \sin(0) + \sin(\frac{\pi}{2}) + \sin(\pi) = 1 \)

2 Summation Properties

- **Constant** \( \sum_{i=1}^{n} c = nc \)
- **Additivity** \( \sum_{i=1}^{n} f(i) + g(i) = \sum_{i=1}^{n} f(i) + \sum_{i=1}^{n} g(i) \)
- **Linearity** \( \sum_{i=1}^{n} af(i) + bg(i) = a \sum_{i=1}^{n} f(i) + b \sum_{i=1}^{n} g(i) \)
- **Constant Multiple** \( \sum_{i=1}^{n} cf(i) = c \sum_{i=1}^{n} f(i) \)
- **Summation Limits** \( \sum_{i=a}^{b} f(i) + \sum_{i=b+1}^{c} f(i) = \sum_{i=a}^{c} f(i) \)
- **Monotonicity** If \( f(i) \leq g(i) \) for each \( i \) then \( \sum_{i=a}^{b} f(i) \leq \sum_{i=a}^{b} g(i) \)

Examples:
1. For \( i \geq 3, i^2 \geq 9 \), therefore \( \sum_{3}^{10} 3 = 3(10 - 2) \leq \sum_{3}^{10} i^2 \)

3 Special Summations

- **Constant** \( \sum_{i=1}^{n} c = nc \)
- \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \)
- \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \)
- \( \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \)

Examples:
1. \( \sum_{1}^{n} 2i - 3i^2 = 2 \sum_{i=1}^{n} i - 3 \sum_{i=1}^{n} i^2 = 2 \frac{n(n+1)}{2} - 3 \frac{n(n+1)(2n+1)}{6} \)
4 Area Computation by Regular Partitions

To find the area of the region bounded by the graph \( y = f(x) \) (with \( f(x) \geq 0 \)), the vertical lines \( x = a \) and \( x = b \) and the x-axis (that is, the area under the curve \( y = f(x) \) between \( a \) and \( b \)), proceed as follows:

1. Subdivide the interval \([a, b]\) into \( n \) subintervals \([x_{i-1}, x_i]\), of equal width \( \Delta x = \frac{b-a}{n} \). The endpoints \( x_i = a + i\Delta x \).

2. In each interval, determine a point \( x_i^* \) by a prescribed method. For example, for circumscribed rectangles choose \( x_i^* \) equal to the point where the absolute maximum of \( f \) occurs in the interval (assuming \( f \) is continuous.)

3. Form the approximation to the area using the Riemann sum,

\[
\sum_{i=1}^{n} f(x_i^*) \Delta x
\]

and simplify using summation formulae.

4. Find the limit as \( n \) “goes to infinity.”

If \( f \) is a continuous function, this limit exists and is called the \textit{definite integral of} \( f \) \textit{from} \( a \) \textit{to} \( b \) and denoted:

\[
\int_{a}^{b} f
\]

5 Exercise

Find the area under the curve \( y = x^2 + x \) from \( x = 1 \) to \( x = 2 \) using the method of regular partitions and circumscribed rectangles.