First NPC Proof: Traveling Salesperson Problem (TSP)

Definition: A graph is complete if there is an edge between every pair of vertices.

![Complete and not complete graphs](image)

**TSP:**

Given: (1) a complete graph \( G = (V, E) \) where each edge \((u,v) \in E\) has cost \(c(u,v) > 0\) and (2) a bound \( k > 0 \)

Goal: Determine whether there is a tour (path that visits every vertex exactly once and returns to origin) with total cost \( \leq k \). \( \Rightarrow \) Decision question: answer is yes/no

**Example:**

![Graph example](image)

\(k = 15\)

Yes: \(d-a-b-c-d\)

\[1\ 2\ 3\ 5\ \text{cost} = 11\]

**Theorem:** TSP is NP-complete

**Proof:** (1) TSP \( \in \) NP

Certificate ? = a path \( S = V_1, V_2, \ldots, V_n, V_1 \)

Poly-time verifiable?
To verify certificate, check:

1) every vertex (except \( v_i \)) appears exactly once. \((S = V - v_i)\).
2) \( v_i \) appears as first & last vertex in \( S \).
3) sum of edge costs \( C_{v_i,v_j} \) in \( S \) \( \leq k \).

\( \checkmark \) can be checked in poly-time \( \checkmark \).

2) \( \text{TSP is NP-hard} \)
   
   For all \( Z \) in \( \text{NP} \), \( Z \in \text{pX} \).

Reduction from \( \text{Ham-Cycle} \)

\( \text{Ham-Cycle} - \) in a general graph is there a cycle that visits every vertex exactly once?

Will show \( \text{Ham-Cycle} \leq_p \text{TSP} \).

If we could solve \( \text{TSP} \), we could solve \( \text{Ham-Cycle} \).

Contra-positive: If we can't solve \( \text{Ham-Cycle} \), can't solve \( \text{TSP} \).
Show that an instance (input) to HC can be converted to an instance of TSP.

\[
\text{instance } (HC) \rightarrow \text{Instance converter} \rightarrow \text{instance (TSP)} \rightarrow \text{Alg(TSP)} \rightarrow \text{sol (TSP)} \rightarrow \text{Solution converter} \rightarrow \text{sol (HC)} \rightarrow \text{(Yes/No)} \rightarrow \text{(Yes/No)}
\]

* All we have to do is find these.

- **Instance Converter first**

Consider the differences b/w an instance of HC and an instance of TSP.

- **Instance of HC**: general (not necessarily complete), unweighted graph
  - TSP: (a) complete graph
  - (b) weights on edges
  - (c) bounded k

Given an instance \( \langle G = (V, E) \rangle \) for HC, we need to convert it to an instance \( \langle G' = (V', E'), w \rangle \) for TSP such that a solution for TSP can be easily converted to a solution for HC.

- A tour of cost \( \leq w \) in \( G' \) indicates a HC in \( G \) (in polytime).
Ex.

\[ G = \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\text{d} \\
\text{e}
\end{array} \quad G' = \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\text{d} \\
\text{e}
\end{array}
\]

(1) \( V' = V \)

(2) \( E' \) has edges to make \( G' \) complete

How to set weights on edges in \( E' \) so that a TSP tour of cost \( \leq k \) is a HC in \( G' \)?

(3) For each edge \((u,v)\) in \( E' \)

- if \((u,v) \in E\) set \( c(u,v) = 1 \)
- else \( c(u,v) = \infty \)

"Which" of these edges will indicate a HC in \( G' \)? Those \( w/cost=\) "How many?" \( |V'| \)

- Cycle that visits every internal node exactly once has exactly \( |V'|-1 \) edges
- (Recall: acyclic path has length at most \( |V'|-1 \))
- Set \( k = |V'| \)

Updates above convert the instance \( \{ \) in poly-time \( \}

Solution converter: \( \text{sol}(HC) = \text{sol}(TSP) \) (= Yes or No)

Ex: \( G' \) has TSP tour \( b-a-c-d-e-b \) with cost \( \leq k = |V'| = 5 \)
\( G ' \) has HC: \( b-a-c-d-e-b \)
**Claim:** $HC \leq P TSP$

There is a TSP tour of cost $\leq k$ in $G'$ if and only if there is a HC in $G$. Why if and only if?

Show: $HC \Rightarrow$ tour and

$\neg HC \Rightarrow \neg$ tour \(\Rightarrow\) tour \(\Rightarrow\) HC.

(\#) A cycle that visits every internal node exactly once has exactly $|V|$ edges.

\[ \implies \text{If there is a TSP tour of cost } \leq k \text{ in } G', \text{ then there is a HC in } G \]

Since the cost of the tour in $G'$ is $|V|$, each edge of the tour must have cost $= 1$, so must correspond to an edge from $G$. These $|V|$ edges will make a HC in $G$.

(\#) \(\Leftarrow\) If there is a HC in $G$, then there is a TSP tour of cost $\leq k = |V|$ in $G'$.

The $|V|$ edges that make the HC in $G$ each have cost $\leq 1$ in $G'$, so these edges will make a TSP tour of cost $= |V|$ in $G'$.

**Note:** There are other correct reductions:

- $Z$ instead of $\infty$
- Set cost of $G$ edges to 0 instead of 1, and
Some questions:

1. Why not show TSP reducible to HC?

   - If we could solve HC we could solve TSP.
   - Does not imply: Since we can't solve HC, can't solve TSP.

   Logic:
   - Solve HC → Solve TSP.
   - Does not imply: If solve HC then solve TSP.

   Showing TSP reducible to HC:

   instance(TSP) → instance(HC) → \( \text{Alg}(Hc) \rightarrow \text{sol}(HC) \rightarrow \text{sol}(TSP) \)

   We know this doesn't exist but possible to solve.

   Still may be possible:

   instance(TSP) → instance(Some Other Problem) → \( \text{Alg} \).

2. Why have to prove iff and not just sol(TSP) = Yes → sol(HC) = Yes?

   Want to show:

   \[ \text{sol}(TSP) = \text{Yes} \rightarrow \text{sol}(HC) = \text{Yes} \quad \text{AND} \quad \text{sol}(TSP) = \text{No} \rightarrow \text{sol}(HC) = \text{No} \rightarrow \neg \text{sol}(TSP) \rightarrow \neg \text{sol}(HC) \]

   \[ \equiv \text{sol}(HC) \rightarrow \text{sol}(TSP). \]